Fast-slow systems with chaotic noise

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Fast-slow systems

Let $\frac{dY}{dt} = g(Y)$ be some 'mildly chaotic' ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \, ,$$

where $\varepsilon \ll 1$ and $h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y)\, \mu(dy) = 0$.

Our aim is to find a **reduced equation** $\frac{d\bar{X}}{dt} = F(\bar{X})$ with $\bar{X} \approx X$.

Fast-slow systems

If we rescale to **large time scales** we have

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$
$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) \ ,$$

We turn $X^{(\varepsilon)}$ into a random variable by taking $Y(0) \sim \mu$. The aim is to characterise the **distribution** of the random path $X^{(\varepsilon)}$ as $\varepsilon \to 0$.

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}) v(Y^{(\varepsilon)}) + f(X^{(\varepsilon)})$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t v(Y^{(\varepsilon)}(s))ds$ then

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

where the integral is of Riemann-Lebesgue type
$(dW^{(\varepsilon)} = \frac{dW^{(\varepsilon)}}{ds} ds)$.

Invariance principle for $W^{(\varepsilon)}$

We can write $W^{(\varepsilon)}$ as

$$W^{(\varepsilon)}(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W^{(\varepsilon)} \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

What about the SDE?

Since

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s))ds$$

But how should we interpret $\star dW$? Stratonovich? Itô? neither?

Continuity with respect to noise (Sussmann '78)

Suppose that

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds\ ,$$

where U is a uniformly continuous path.

If $h(x) \equiv Id$ **or** $n = d = 1$, then the above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

The simple case (Melbourne, Stuart '11)

If the flow is chaotic enough so that

$$W^{(\varepsilon)} \Rightarrow W\ ,$$

and $h \equiv \mathrm{Id}$ **or** $n = d = 1$

then we have that $X^{(\varepsilon)} \Rightarrow X$ in the sup-norm topology, where

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds\ ,$$

where the stochastic integral is of Stratonovich type.

Continuity of the solution map

The solution map takes "noisy path space" to "solution space"

$$\Phi : W^{(\varepsilon)} \mapsto X^{(\varepsilon)}$$

If this map were **continuous** then we could lift $W^{(\varepsilon)} \Rightarrow W$ to $X^{(\varepsilon)} \Rightarrow X$.

When the noise is both **multidimensional** and **multiplicative**, this strategy fails.

Continuity of the solution map

We want to define a map $\Phi : U \to X$ where U is a noisy path and

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds$$

This is problematic for two reasons.

1 - The solution map Φ is only defined for *differentiable* noise. But noisy paths like Brownian motion are **not differentiable** (they are *almost* $1/2$-Hölder).

2 - Any attempt to define an extension of Φ to Brownian-like objects will **fail to be continuous**. ie. We can find a sequence $W_n \to W$ but $\Phi(W_n) \not\to \Phi(W)$.

To build a continuous solution map, we need **extra information** about U.

Rough path theory (Lyons '97)

Suppose we are given a path $\mathbb{U} : [0, T] \to \mathbb{R}^{d\times d}$ which is (formally) an iterated integral

$$\mathbb{U}^{ij}(t) \stackrel{def}{=} \int_0^t U^i(s) dU^j(s) \;.$$

Given a "rough path" $\mathbf{U} = (U, \mathbb{U})$ we can construct a solution

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{U}(s) + \int_0^t h(X(s)) ds$$

The map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an extension of the classical solution map and is **continuous** with respect to the "rough path topology".

Convergence of fast-slow systems

If we let

$$\mathbb{W}^{ij,(\varepsilon)}(t) = \int_0^t W^{i,(\varepsilon)}(r)dW^{j,(\varepsilon)}(r)$$

then $X^{(\varepsilon)} = \Phi(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)})$.

Due to the continuity of Φ, if $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$, then $X^{(\varepsilon)} \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

We have the following result

Theorem (K. & Melbourne '14)

If the fast dynamics are "sufficiently chaotic", then $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s) dW^j(s) + \lambda^{ij} t$$

where the integral is Itô type and

$$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds \,.$$

$$\mathrm{Cov}^{ij}(W) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0)) v^j(Y(s)) + v^j(Y(0))\, v^i(Y(s)) \} \, ds$$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X^{(\varepsilon)} \Rightarrow \bar{X}$ *where*

$$d\bar{X} = h(\bar{X})dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt \; .$$

in Itô form, with λ^{ij} " = " $\int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds$

$$d\bar{X} = h(\bar{X}) \circ dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt$$

in Stratonovich form, with
λ^{ij} " = " $\int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) - v^j(Y(0))\, v^i(Y(s)) \}\, ds$.

General fast-slow systems I

What about the original (much more complicated) fast-slow system?

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$

$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) \ .$$

General fast-slow systems II

The slow variables

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) ds + \int_0^t f(X^{(\varepsilon)}, Y^{(\varepsilon)}) ds$$

can be written in the **product form**

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t H(X^{(\varepsilon)}(s)) dW^{(\varepsilon)}(s) + \int_0^t H(X^{(\varepsilon)}(s)) dV^{(\varepsilon)}(s)$$

H is the **evaluation map** (or Dirac distribution) $H(x)\varphi = \varphi(x)$ for $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ suitably smooth. And $W^{(\varepsilon)}, V^{(\varepsilon)}$ are the **function valued paths**

$$W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t h(\cdot, Y^{(\varepsilon)}(s)) ds \quad V^{(\varepsilon)}(t) = \int_0^t f(\cdot, Y^{(\varepsilon)}(s)) ds$$

General fast-slow systems III

Theorem (K. & Melbourne '14)

If the fast dynamics are "sufficiently chaotic" then $X^{(\varepsilon)} \Rightarrow \bar{X}$ *where*

$$d\bar{X} = \sigma(\bar{X})dB + \tilde{a}(\bar{X})dt \,,$$

where B *is a standard BM on* $\mathbb{R}^d$ *and*

$$\tilde{a}(x) = \int f(x,y)d\mu(y) + \sum_{k=1}^{d} \mathfrak{B}(h^k(x,\cdot), \partial_k h(x,\cdot))$$

$$\sigma\sigma^T(x) = \mathfrak{B}(h^i(x,\cdot), h^j(x,\cdot)) + \mathfrak{B}(h^j(x,\cdot), h^i(x,\cdot))$$

and $\mathfrak{B}$ *is the "integrated autocorrelation" of the fast dynamics*

$$\mathfrak{B}(v,w) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu v(Y(0))w(Y(s))ds$$

The real world has feedback

It is more realistic to look fast-slow systems of the form

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$
$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) + \varepsilon^{\beta-2} g_0(X^{(\varepsilon)}, Y^{(\varepsilon)}) \ ,$$

for some $\beta \geq 1$. Since the coupling term is of lower order, this is called **weak feedback**.

Back of the envelope: For $\beta > 1$, the reduced model is exactly the same as the the zero feedback case.
For $\beta = 1$, an additional correction term appears, which involves the weak feedback term g_0.

The real world is infinite dimensional

Many fast-slow models are **PDEs**.

Suppose that $Y^{(\varepsilon)} = (Y_1^{(\varepsilon)}, Y_2^{(\varepsilon)}, \dots)$ is an infinite vector of fast, chaotic variables (possibly coupled). Can we identify a reduced model for $X^{(\varepsilon)} = X^{(\varepsilon)}(t, x)$ where

$$\partial_t X^{(\varepsilon)} = \Delta X^{(\varepsilon)} + \varepsilon^{-1} H(X^{(\varepsilon)}, Y^{(\varepsilon)}) + F(X^{(\varepsilon)}, Y^{(\varepsilon)})$$

This is a delicate question, since many natural approximations of noise yield **infinites** in the limiting SPDE.

This is a problem for Hairer's theory of **regularity structures**.

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. To appear in **Ann. Probab.** (2014).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2014).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. To appear in **Ann. App. Probab.** (2014).

All my slides are on my website (www.dtbkelly.com) **Thank you**!