Fast-slow systems with chaotic noise

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Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \,,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y)\, \mu(dy) = 0$.

Our aim is to find a **reduced equation** $\frac{d\bar{X}}{dt} = F(\bar{X})$ with $\bar{X} \approx X$.

Fast-slow systems

If we rescale to **large time scales** we have

$$\begin{aligned}\frac{dX_\varepsilon}{dt} &= \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)\\ \frac{dY_\varepsilon}{dt} &= \varepsilon^{-2} g(Y_\varepsilon)\ ,\end{aligned}$$

We turn X_ε into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path X_ε as $\varepsilon \to 0$.

Why is model reduction important?

1 - The reduced model is **lower dimensional** and **less stiff** than the original fast-slow system.

2 - Helps the user make **informed guess** when the model is unknown.

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon) v(Y_\varepsilon) + f(X_\varepsilon)$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(Y_\varepsilon(s))ds$ then

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

where the integral is of Riemann-Lebesgue type ($dW_\varepsilon = \frac{dW_\varepsilon}{ds}ds$).

Invariance principle for W_ε

We can write W_ε as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

We will call this class of Y **mildly chaotic**.

What about the SDE?

Since

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s)) ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s)) ds$$

But how should we interpret $\star dW$? Stratonovich? Itô? neither?

For **additive noise** $h(x) = I$
the answer is simple.

Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \ ,$$

where U is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

The simple case (Melbourne, Stuart ‘11)

If the flow is mildly chaotic ($W_\varepsilon \Rightarrow W$) then $X_\varepsilon \Rightarrow \bar{X}$ in the sup-norm topology, where

$$d\bar{X} = dW + f(\bar{X})ds\ .$$

Same idea works for general h provided $n = d = 1$. The limit is Stratonovich

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds\ .$$

The strategy

The solution map takes “noisy path space” to “solution space”

$$\Phi : W_\varepsilon \mapsto X_\varepsilon$$

If this map were **continuous** then we could lift $W_\varepsilon \Rightarrow W$ to $X_\varepsilon \Rightarrow X$.

When the noise is both **multidimensional** and **multiplicative**, this strategy fails.

Ito, Stratonovich and family

SDEs are very **sensitive** wrt approximations of BM.

Suppose

$$dX = h(X)dW + f(X)dt$$

and define an approximation

$$dX_n = h(X_n)dW_n + f(X_n)dt$$

with some approximation W_n of W.

Taking $n \to \infty$, X_n might converge to something completely different to X. It all depends on the approximation W_n.

Eg. 1 If W_n is a step function approximation of W, then X_n converges to the **Ito** SDE

$$dX = h(X)dW + f(X)dt$$

Eg. 2 (Wong-Zakai) If W_n is a linear interpolation of W, then X_n converges to the **Stratonovich** SDE

$$dX = h(X) \circ dW + f(X)dt$$

Eg. 3 (Sussman) If W_n is a higher order spline interpolation of W, we can get limits which are **neither Ito nor Stratonovich**.

It is not enough to know that $W_n \to BM$.

We need more information.

Rough path theory (Lyons '97)

Provides a **unified** definition of a DE driven by a noisy path

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t h(X(s))ds$$

when the dU integral is not well defined.

In addition to U we must be given another path $\mathbb{U} : [0, T] \to \mathbb{R}^{d\times d}$ which is (formally) an iterated integral

$$\mathbb{U}^{ij}(t) \overset{def}{=} \int_0^t U^i(s)dU^j(s) \, .$$

These extra components tells us how to interpret the **method of integration**.

Rough path theory (Lyons ‘97)

Given a “rough path” $\mathbf{U} = (U, \mathbb{U})$ we can construct a solution

$$X(t) = X(0) + \int_0^t h(X(s))d\mathbf{U}(s) + \int_0^t h(X(s))ds$$

Eg. 1 If $U = W$ and $\mathbb{U} = \int W dW$ is the Ito iterated integral, then the constructed X is the solution to the Ito SDE.

Eg. 2 If $U = W$ and $\mathbb{U} = \int W \circ dW$ is the Stratonovich iterated integral, then the constructed X is the solution to the Stratonovich SDE.

Rough path theory (Lyons '97)

Most importantly (for us) the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an extension of the classical solution map and is **continuous** with respect to the "rough path topology".

Convergence of fast-slow systems

Returning to the slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

If we let

$$\mathbb{W}_\varepsilon^{ij}(t) = \int_0^t W_\varepsilon^i(r)dW_\varepsilon^j(r)$$

then $X_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon)$.

Due to the continuity of Φ, if $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$, then $X_\varepsilon \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

Theorem (K. & Melbourne '14)

If the fast dynamics are mildly chaotic, then $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s)dW^j(s) + \lambda^{ij} t$$

where the integral is Itô type and

$$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu\{v^i(Y(0))\, v^j(Y(s))\}\, ds\ .$$

$$\mathrm{Cov}^{ij}(W) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu\{v^i(Y(0)) v^j(Y(s)) + v^j(Y(0))\, v^i(Y(s)))\}\, ds$$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X_\varepsilon \Rightarrow \bar{X}$ *where*

$$d\bar{X} = h(\bar{X})dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt \ .$$

in Itô form, with $\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds$

$$d\bar{X} = h(\bar{X}) \circ dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt$$

in Stratonovich form, with
$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) - v^j(Y(0))\, v^i(Y(s)) \}\, ds$.

General fast-slow systems I

What about the original (much more complicated) fast-slow system?

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$

$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \, .$$

General fast-slow systems II

The slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) ds + \int_0^t f(X_\varepsilon, Y_\varepsilon) ds$$

can be written in the **product form**

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t H(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t H(X_\varepsilon(s)) dV_\varepsilon(s)$$

H is the **evaluation map** (or Dirac distribution) $H(x)\varphi = \varphi(x)$ for $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ suitably smooth. And $W_\varepsilon, V_\varepsilon$ are the **function valued paths**

$$W_\varepsilon(t) = \varepsilon^{-1} \int_0^t h(\cdot, Y_\varepsilon(s)) ds \quad V_\varepsilon(t) = \int_0^t f(\cdot, Y_\varepsilon(s)) ds$$

General fast-slow systems III

Theorem (K. & Melbourne '14)

If the fast dynamics are mildly chaotic then $X_\varepsilon \Rightarrow \bar{X}$ *where*

$$d\bar{X} = \sigma(\bar{X})dB + \tilde{a}(\bar{X})dt \; ,$$

where B *is a standard BM on* $\mathbb{R}^d$ *and*

$$\tilde{a}(x) = \int f(x,y)d\mu(y) + \sum_{k=1}^{d} \mathfrak{B}(h^k(x,\cdot), \partial_k h(x,\cdot))$$

$$\sigma\sigma^T(x) = \mathfrak{B}(h^i(x,\cdot), h^j(x,\cdot)) + \mathfrak{B}(h^j(x,\cdot), h^i(x,\cdot))$$

and $\mathfrak{B}$ *is the "integrated autocorrelation" of the fast dynamics*

$$\mathfrak{B}(v,w) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu v(Y(0))w(Y(s))ds$$

The real world has feedback

It is more realistic to look fast-slow systems of the form

$$\begin{aligned} \frac{dX_\varepsilon}{dt} &= \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon) \\ \frac{dY_\varepsilon}{dt} &= \varepsilon^{-2} g(Y_\varepsilon) + \varepsilon^{\beta-2} g_0(X_\varepsilon, Y_\varepsilon)\ , \end{aligned}$$

for some $\beta \geq 1$. Since the coupling term is of lower order, this is called **weak feedback**.

Back of the envelope: For $\beta > 1$, the reduced model is exactly the same as the the zero feedback case.
For $\beta = 1$, an additional correction term appears, which involves the weak feedback term g_0.

The real world is infinite dimensional

Many fast-slow models are **PDEs**.

Suppose that $Y_\varepsilon = (Y_\varepsilon^1, Y_\varepsilon^2, \dots)$ is an infinite vector of fast, chaotic variables (possibly coupled). Can we identify a reduced model for $X_\varepsilon = X_\varepsilon(t, x)$ where

$$\partial_t X_\varepsilon = \Delta X_\varepsilon + \varepsilon^{-1} H(X_\varepsilon, Y_\varepsilon) + F(X_\varepsilon, Y_\varepsilon)$$

This is a delicate question, since many natural approximations of noise yield **infinites** in the limiting SPDE.

This is a problem for Hairer's theory of **regularity structures**.

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. To appear in **Ann. Probab.** (2014).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2014).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. To appear in **Ann. App. Probab.** (2014).

All my slides are on my website (www.dtbkelly.com) **Thank you**!