Stochastic Modelling and ENSO

David Kelly

Mathematics Department
UNC Chapel Hill

dtbkelly@gmail.com

December 3, 2013

Outline

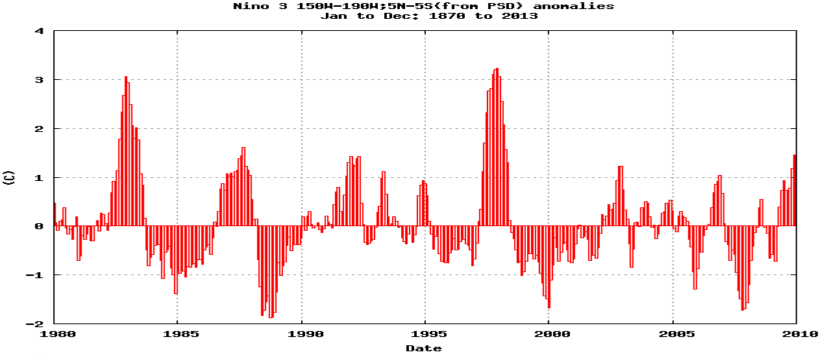
1 - Why is ENSO **stochastic**?

2 - **Variability** can be explained by stochastic noise (Kleeman + Moore paper)

3 - Chaotic models **vs** stochastic models

What are the stochastic features of ENSO?

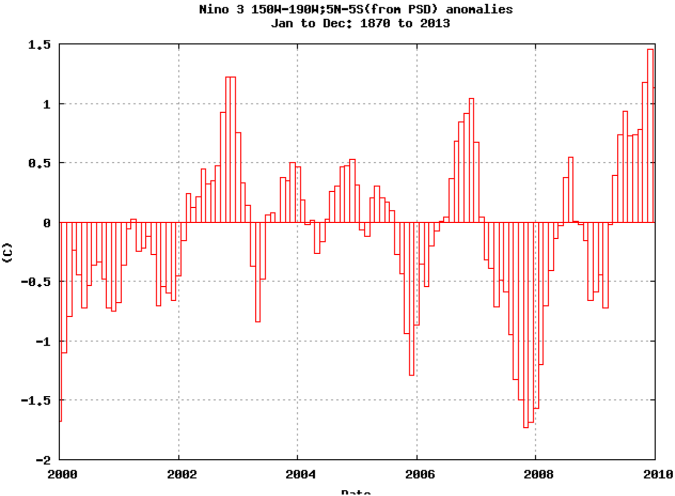
Variability of Amplitude and Period



Data file

- No two events are the same in magnitude.
- Is it really an oscillation? $2 - 10$ yr periods.

Seasonal locking



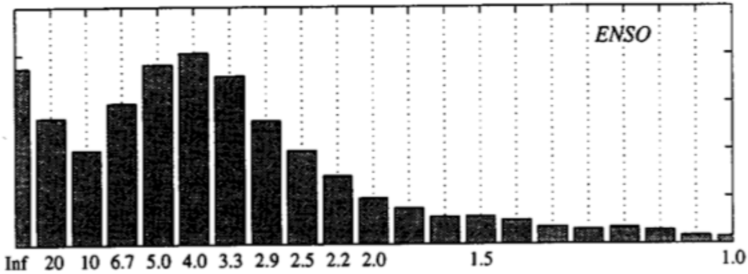
- However, peaks tend to happen around December.

Spectral Analysis

The spectral density indicates the dominant modes of a signal.

For a random signal $f(t)$, the spectral density is given by $\mathbf{E}|\hat{f}(k)|^2$ where $\hat{f}$ is the Fourier transform of f.
Eg.

Spectral Analysis



- 4 year period is significant ... But is surrounded by a lot of "noise".
- The spectrum decays like k^{-2} ... Signature of **red noise** (Brownian motion).

Why is ENSO stochastic

The ENSO phenomenon features **variability** in

- amplitude of events
- frequency of events

and has a spectral signature reminiscent of **noise**.

This suggests noise is **present** ... But we would like to know if the noise is actually "**causing**" excitations.

Evidence that ENSO events arise due to noise.

Kleeman and Moore. *A Theory of the Limitation of ENSO Predictability Due to Stochastic Atmospheric Transients*. **Journal of Atmospheric Science** (1997).

Kleeman and Moore. *Stochastic Forcing of ENSO by the Intraseasonal Oscillation*. **Journal of Climate** (1999).

Idea behind paper

1 - Fix the model dynamics $\boldsymbol{\Psi}$

2 - Propose a noisy perturbation ψ

3 - Find the type of noise that excites variability

4 - Show that this agrees with natural sources of noise

The model

The authors used a coupled atmosphere-ocean model ($\star$) (variant of the ZC model in Dijkstra).

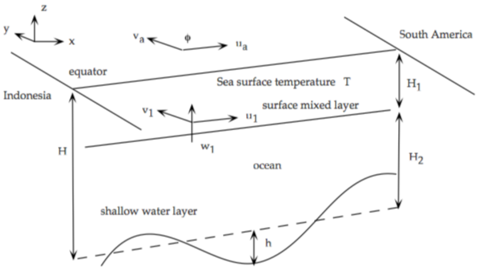


Figure 7.18. Schematic representation of the Zebiak-Cane model showing both surface layer and shallow water layer. the latter bounded below by the thermocline.

($\star$) - Kleeman (1991,1993).

The model

The noise is assumed to enter the model through forcing in the **wind components**.

This is reasonable given short term, unpredictable wind events, like **Madden-Julian oscillation** (MJO).

Linear stability analysis

Suppose that $\Psi : \mathbb{R}_+ \to \mathbb{R}^N$ is the interannual climate variables

$$\frac{d\Psi}{dt} = F(\Psi)$$

given by a spatially discretized version of the model.

Let ψ be the anomaly, and suppose that

$$\frac{d(\Psi + \psi)}{dt} = F(\Psi + \psi) + f(t)$$

where f is an undefined source of noise.

Linear stability analysis

If we assume that $\psi^2 \ll \psi$ then

$$\frac{d(\Psi + \psi)}{dt} = F(\Psi) + \nabla F(\Psi)\psi + O(\psi^2) + f(t)$$

This gives the linear approximation

$$\frac{d\psi}{dt} = \nabla F(\Psi)\psi + f(t)$$

So we have a linear model for the anomaly ψ that depends on the state of the unperturbed model Ψ.

Discretization of the model

We work with the time discretization

$$\psi_{n+1} = \psi_n + F_n \psi_n \Delta t + f_n \Delta t \ .$$

where $F_n = \nabla F(\Psi_n)$.

We let $f_n = \Delta t^{-1/2} \xi_n$, where ξ is a sequence of identically distributed Gaussian random variables, with

$$\mathbf{E} \xi_n = 0 \quad \text{and} \quad \mathbf{E} \xi_n \xi_m^T = D_{n,m} C \ .$$

The number $D_{n,m}$ measures temporal correlation and the matrix C measures spatial correlation.

Writing down the solution

Since the model is linear, the solution is **easy to write down**. Let $R_{j,k}$ be the semi-group for the linear part. That is, if

$$u_{n+1} = u_n + A_n u_n \Delta t$$

then $u_k = R_{j,k} u_j$. We will solve

$$\psi_{n+1} = \psi_n + A_n \psi_n \Delta t + \xi_n \Delta t^{1/2}$$

using **Duhamel's principle**.

Writing down the solution

We have that ...

$$
\begin{aligned}
\psi_{n+1} &= \psi_n + A_n \psi_n \Delta t + \xi_n \Delta t^{1/2} \\
&= (1 + A_n \Delta t)\psi_n + \xi_n \Delta t^{1/2} \\
&= R_{n,n+1}\psi_n + \xi_n \Delta t^{1/2}
\end{aligned}
$$

Repeating this ...

$$
\begin{aligned}
\psi_{n+1} &= R_{n,n+1}(R_{n-1,n}\psi_{n-1} + \xi_{n-1}\Delta t^{1/2}) + \xi_n \Delta t^{1/2} \\
&= R_{n-1,n+1}\psi_{n-1} + \left(R_{n-1,n}\xi_{n-1}\Delta t^{1/2} + R_{n,n}\xi_n \Delta t^{1/2} \right)
\end{aligned}
$$

And finally

$$
\psi_{n+1} = R_{0,n+1}\psi_0 + \sum_{k=0}^{n} R_{k,n}\xi_k \Delta t^{1/2}
$$

Measuring the variability of the anomaly

Given the solution, it is easy to write down the mean

$$\mathbf{E}\psi_n = \mathbf{E}\left(R_{0,n}\psi_0 + \sum_{k=0}^{n-1} R_{k,n-1}\xi_k \Delta t^{1/2} \right) = R_{0,n}\mathbf{E}\psi_0 \ .$$

And the variance $\mathbf{E}\left(|\psi_n - \mathbf{E}\psi_n|^2 \right)$ is given by

$$\mathbf{E}\langle R_{0,n}(\psi_0 - \mathbf{E}\psi_0), R_{0,n}(\psi_0 - \mathbf{E}\psi_0)\rangle + \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathbf{E}\langle R_{k,n-1}\xi_k, R_{j,n-1}\xi_j\rangle \Delta t$$

Measuring the variability of the anomaly

The **noise** term is the important one. It simplifies to

$$\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathbf{E}\langle R_{k,n-1}\xi_k , R_{j,n-1}\xi_j\rangle \Delta t = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathbf{E}\langle R_{j,n-1}^T R_{k,n-1}\xi_k , \xi_j\rangle \Delta t$$

$$= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} tr\left(R_{j,n-1}^T R_{k,n-1} D_{j,k} C \right)\Delta t = tr(ZC)$$

where

$$Z = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} R_{j,n-1}^T R_{k,n-1} D_{j,k} \Delta t$$

Measuring the variability of the anomaly

Alternatively, we might want to measure the anomaly of a particular **feature** of the model.

Let $P : \mathbb{R}^N \to \mathbb{R}^M$ for some $M \leq N$.

Eg. P could be the NINO3 average.

Then

$$\mathbf{E}|P(\psi_n - \mathbf{E}\psi_n)|^2 = tr(ZC)$$

where

$$Z = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} R_{j,n-1}^T P^T P R_{k,n-1} D_{j,k} \Delta t$$

NB. From now on we **always** use the NINO3 average for P

Stochastic Optimals

Let $\{v_k, \lambda_k\}$ and $\{w_k, \mu_k\}$ be the eigenvectors-eigenvalue pairs for Z and C respectively. Then

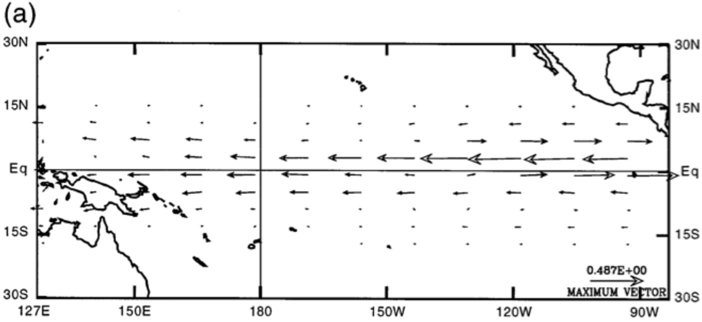
$$tr(ZC) = \sum_{i,j=1}^{N} \lambda_i \mu_j |\langle v_i, w_j \rangle|^2 \,.$$

The eigenvectors of Z are called the **stochastic optimals**.

The eigenvectors of C are called the **empirical orthogonal functions** (EOFs).

The anomaly is activated when the dominant stochastic optimal lines up with the dominant EOF.

Stochastic Optimals: Wind stress



First stochastic optimal (wind stress component).

We can compare the dominant stochastic optimal for wind stress, with the dominant eigenvector of the observed wind stress.

Observed wind stress

1 - Get a data set of wind-stress observations $\{W_1, \dots, W_M\}$.

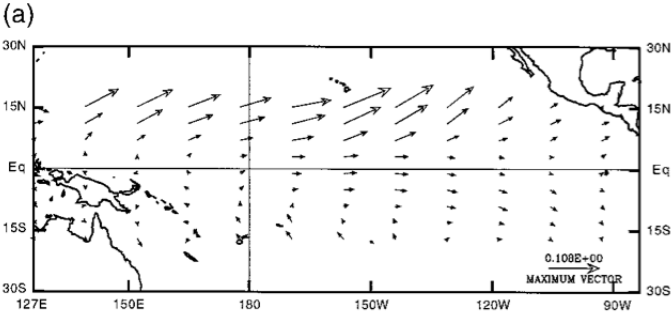
2 - Filter out the large time scales to obtain $\{\tilde{W}_1, \dots, \tilde{W}_M\}$.

3 - Compute the covariance matrix

$$C = \frac{1}{N} \sum_{j=1}^{M} (\tilde{W}_j - \bar{W})(\tilde{W}_j - \bar{W})^T$$

4 - Find the eigenvectors (EOFs) of C.

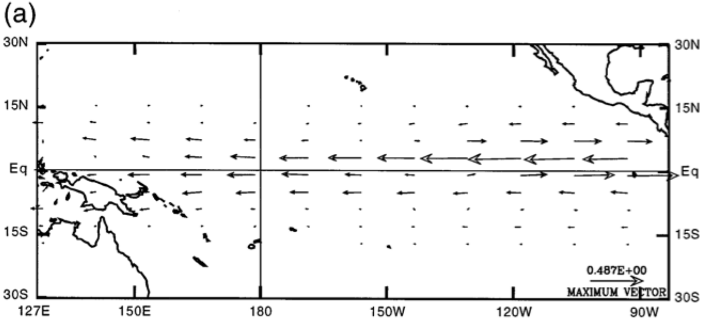
Observed wind stress



First eigenvector of the covariance.

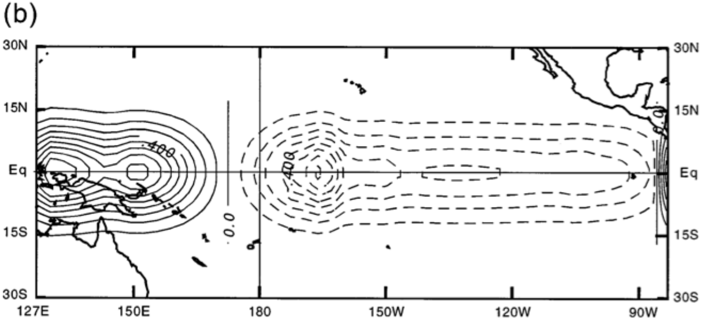
- Similar global pattern (up to plus-minus).

Stochastic Optimals: Wind stress



First stochastic optimal (wind stress component).

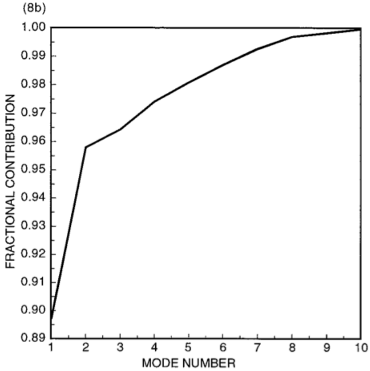
Stochastic Optimals: Heat flux



- The dipole structure indicates the importance of coupling in ENSO events.
- Dipole structures agrees with MJO heat flux map.

The first stochastic optimal / EOF pair accounts for almost all of the anomaly.

Importance of stochastic optimals



Truncated version of the variance $tr(ZC) = \sum_{i,j=1}^{N} \lambda_i \mu_j |\langle v_i, w_j \rangle|^2$

What do simulations look like?

JOURNAL OF CLIMATE

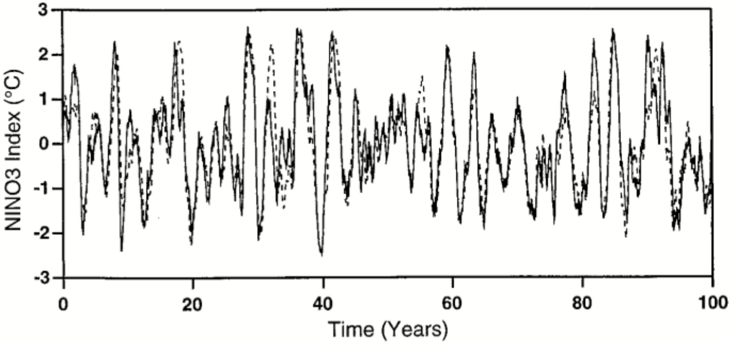


FIG. 12. Time series of the NINO3 index from 100-yr integrations of the coupled model forced with stochastic noise composed of $\mathbf{S}_1$ and $\mathbf{S}_2$. Solid curve shows the case where only the surface heat flux components of $\mathbf{S}_1$ and $\mathbf{S}_2$ are used, and the dashed curve shows the case where only the surface wind stress components are used.

Seasonal locking of extreme events

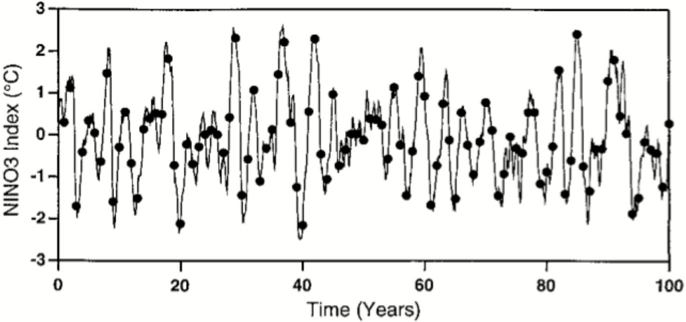


FIG. 2. A time series of the NINO3 index from a 100-yr integration of the coupled model forced with stochastic noise composed of $\mathbf{S}_1$ and $\mathbf{S}_2$. The bullets indicate 1 Dec of each year.

Spectral analysis

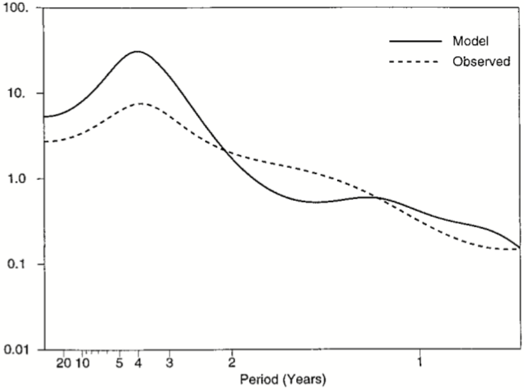


FIG. 3. A comparison of the power spectrum of the NINO3 index from the stochastically forced coupled model and observation. The spectra were computed using a maximum entropy method of order 30.

Chaotic forcing vs stochastic forcing.

Slow-Fast system

Suppose the weather variables y^ε satisfy some chaotic dynamics

$$\frac{dy^\varepsilon}{dt} = \varepsilon^{-2} g(y^\varepsilon)$$

Note that $y^\varepsilon(t) = y(\varepsilon^{-2} t)$ where $\dot{y} = g(y)$.

Suppose the climate variables x satisfy

$$\frac{dx^\varepsilon}{dt} = \varepsilon^{-1} h(x^\varepsilon, y^\varepsilon) + f(x^\varepsilon, y^\varepsilon)$$

This is a natural set-up in climate models.
Eg. Barotropic flow (Majda et al 1999).

Slow-Fast system

For mathematical convenience, we assume that $h = h(y)$ and $f = f(x)$, so that

$$\frac{dx^\varepsilon}{dt} = \varepsilon^{-1} h(y^\varepsilon) + f(x^\varepsilon)$$

Or in the integral form

$$x^\varepsilon(t) = x^\varepsilon(0) + \varepsilon^{-1} \int_0^t h(y^\varepsilon(s))ds + \int_0^t f(x^\varepsilon(s))ds$$

The fast dynamics

When $\varepsilon \ll 1$, the fast component behaves very **randomly** - just like the Bernoulli shift. If we assume that the initial condition of y^ε is distributed randomly, then

$$W^\varepsilon(t) = \varepsilon^{-1} \int_0^t h(y^\varepsilon(s))ds$$

becomes a random variable.

If we can **classify** the **statistics** of W^ε in the limit, then perhaps we can do the same for x^ε.

The fast dynamics

Since y^ε behaves randomly, the signal W^ε is the sum of a sequence of decorrelated random variables.

$$\varepsilon^{-1}\int_0^t h(y^\varepsilon(s))ds = \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor} \varepsilon^{-1}\int_{j\varepsilon^2}^{(j+1)\varepsilon^2} h(y^\varepsilon(s))ds$$
$$= \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor} \varepsilon\int_{j}^{j+1} h(y(s))ds$$

Recall that this is how to build **Brownian motion**.

One can actually show that the statistics of W^ε converge to the statistics of (a multiple of) Brownian motion B as $\varepsilon \to 0$.

A continuous map

In general, the map $Z \mapsto x$ defined by the solution to

$$x(t) = x(0) + Z(t) + \int_0^t f(x(s))ds$$

is **continuous** in the sup-norm topology.

This means that if convergence results for Z translate nicely to convergence results for x.

Convergence to an SDE

So if W converges to B, then the solution of

$$x^{\varepsilon}(t) = x^{\varepsilon}(0) + W^{\varepsilon}(t) + \int_0^t f(x^{\varepsilon}(s))ds$$

converges to

$$x(t) = x(0) + B(t) + \int_0^t f(x(s))ds$$

Or as an SDE

$$dx = dB + F(x)dt$$

General systems

The same type of result holds for the general slow-fast system

$$\frac{dx^\varepsilon}{dt} = \varepsilon^{-1} h(x^\varepsilon, y^\varepsilon) + f(x^\varepsilon, y^\varepsilon)$$

but the argument is a lot more complicated.

The **statistical behaviour** of a deterministic, chaotic slow-fast system can be approximated by an **SDE**.