Scientific Computing: Ordinary Differential Equations

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Outline

Initial Value Problems

- We want to numerically approximate the solution to the **ordinary differential equation**

$$\frac{dx}{dt} = x'(t) = \dot{x}(t) = f\left(x(t), t\right),$$

with **initial condition** $x(t = 0) = x(0) = x_0$.

- This means that we want to generate an approximation to the **trajectory** $x(t)$, for example, a sequence $x(t_k = k\Delta t)$ for $k = 1, 2, \ldots, N = T/\Delta t$, where Δt is the **time step** used to discretize time.
- If f is independent of t we call the system **autonomous**.
- Note that second-order equations can be written as a **system** of first-order equations:

$$\frac{d^2x}{dt^2} = \ddot{x}(t) = f\left[x(t), t\right] \quad \equiv \quad \begin{cases} \dot{x}(t) = & v(t) \\ \dot{v}(t) = & f\left[x(t), t\right] \end{cases}$$

Relation to Numerical Integration

- If f is independent of x then the problem is equivalent to numerical integration

$$x(t) = x_0 + \int_0^t f(s)ds.$$

- More generally, we cannot compute the integral because it depends on the unknown answer $x(t)$:

$$x(t) = x_0 + \int_0^t f\left(x(s), s\right) ds.$$

- Numerical methods are based on approximations of $f\left(x(s), s\right)$ into the "future" based on knowledge of $x(t)$ in the "past" and "present".

Convergence

- Consider a trajectory numerically discretized as a sequence that **approximates** the exact solution at a **discrete** set of points:

$$x^{(k)} \approx x(t_k = k\Delta t), \quad k = 1, \ldots, T/\Delta t.$$

- A method is said to **converge with order** $p > 0$, or to have **order of accuracy** p, if for any finite T for which the ODE has a solution,

$$\left| x^{(k)} - x(k\Delta t) \right| = O(\Delta t^p) \text{ for all } 0 \leq k \leq T/\Delta t.$$

- All methods are recursions that compute a new $x^{(k+1)}$ from previous $x^{(k)}$ by evaluating $f(x)$ several times. For example, **one-step methods** have the form

$$x^{(k+1)} = G\left(x^{(k)}, \Delta t;\, f\right).$$

Consistency

- The **local trunction error** (LTE) of a method is the amount by which the *exact* solution does not satisfy the numerical scheme at the end of the time step if started from the correct solution $x^{(k)} = x\left(k\Delta t\right)$:

$$e_k = x\left[\left(k+1\right)\Delta t\right] - G\left[x(k\Delta t), \Delta t;\, f\right],$$

- A method is **consistent with order** $q > 1$ if $|e_k| = O(\Delta t^q)$.
- The **global truncation error** is the actual error

$$E_{t=k\Delta t} = \left|x^{(k)} - x(t = k\Delta t)\right|.$$

- Numerical analysis question: Can the global error be bounded by $O\left(\Delta t^p\right)$ if the local one is $O(\Delta t^q)$?

Propagation of errors

- *Crude* estimate: If one makes an error $O(\Delta t^q)$ at each time step, the global error after $T/\Delta t$ time steps can become on the order of

$$\left|x^{(k)} - x(k\Delta t)\right| = O\left(\Delta t^q \cdot \frac{T}{\Delta t}\right) = O\left(\Delta t^{q-1}\right) = O(\Delta t^p),$$

  and we must have $p = q - 1 > 0$ for convergence.
- This result is often the right one, but it has a hidden assumption that *errors made at previous steps do not grow* but rather stay of the same order so they can be added.
- In practice, errors made in previous time steps will either grow or shrink with time. If they grow "too fast" we are in trouble.
- So we arrive for the first time at a recurring theme: **Convergence requires stability in addition to consistency**. We discuss stability later on, after we give some basic methods for solving ODEs.

Euler's Method

- Assume that we have our approximation $x^{(k)}$ and want to move by one time step:

$$x^{(k+1)} \approx x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left(x(s), s\right) ds.$$

- The simplest possible thing is to use a piecewise constant approximation:

$$f\left(x(s), s\right) \approx f(x^{(k)}) = f^{(k)},$$

which gives the **forward Euler method**

$$x^{(k+1)} = x^{(k)} + f^{(k)} \Delta t.$$

- This method requires only one function evaluation per time step.

Euler's Method

$$\text{Scheme: } x^{(k+1)} - x^{(k)} - f^{(k)}\Delta t = 0$$

- The local trunction error is easy to find using a Taylor series expansion:

$$\begin{aligned} e_k &= x\left[(k+1)\,\Delta t\right] - x\left(k\Delta t\right) - f\left[x\left(k\Delta t\right)\right]\Delta t = \\ &= x\left[(k+1)\,\Delta t\right] - x\left(k\Delta t\right) - \left[x'\left(k\Delta t\right)\right]\Delta t = \frac{x''(\xi)}{2}\Delta t^2, \end{aligned}$$

  for some $k\Delta t \le \xi \le (k+1)\,\Delta t$.
- Therefore the LTE is $O(\Delta t^2)$, $q = 2$.
- The global truncation error, however, is of order $O(\Delta t)$, $p = q + 1$, so this is a **first-order accurate** method.

Backward Euler

$$x^{(k+1)} \approx x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left[x(s), s\right] ds.$$

- How about we use a piecewise constant-approximation, but based on the end-point:

  $$f\left[x(s), s\right] \approx f(x^{(k+1)}) = f^{(k+1)},$$

  which gives the first-order **backward Euler method**

  $$x^{(k+1)} = x^{(k)} + f(x^{(k+1)})\Delta t.$$

- This **implicit method** requires **solving a non-linear equation** at every time step, which is expensive and hard.
  We will understand why implicit methods are needed when we examine absolute stability later on.

Runge-Kutta Methods

- **Runge-Kutta methods** are a powerful class of **one-step methods** similar to Euler's method, but more accurate.
- As an example, consider using a trapezoidal rule to approximate the integral

$$x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left[x(s), s\right] ds \approx x^{(k)} + \frac{\Delta t}{2}\left[f\left(k\Delta t\right) + f\left(\left(k+1\right)\Delta t\right)\right],$$

$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{2}\left[f\left(x^{(k)},\ t^{(k)}\right) + f\left(x^{(k+1)},\ t^{(k+1)}\right)\right]$$

  which requires solving a nonlinear equation for $x^{(k+1)}$.
- This is the simplest **implicit Runge-Kutta method**, usually called the **implicit trapezoidal method**.
- The local truncation error is $O(\Delta t^3)$, so the global error is **second-order accurate** $O(\Delta t^2)$.

Midpoint/Trapezoidal Methods

- Schemes that **treat beginning and end of time step in a symmetric fashion** will lead to a **cancellation of first-order error terms** in Taylor series and will thus be **second order** (Lesson: second order is easy).
- In addition to trapezoidal one can do **implicit midpoint** scheme:

$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{2} f\left(\frac{x^{(k)} + x^{(k+1)}}{2},\ t^{(k)} + \frac{\Delta t}{2}\right)$$

  Observe this is the same as trapezoidal for linear problems (why?).
- In an explicit method, we would approximate $x^\star \approx x^{(k+1)}$ first using Euler's method, to get the simplest **explicit Runge-Kutta method**, usually called **Heun's or explicit trapezoidal method**

$$\begin{aligned} x^{(k+1,\star)} &= x^{(k)} + f\left(x^{(k)},\ t^{(k)}\right)\Delta t \\ x^{(k+1)} &= x^{(k)} + \frac{\Delta t}{2}\left[f\left(x^{(k)},\ t^{(k)}\right) + f\left(x^{(k+1,\star)},\ t^{(k+1)}\right)\right]. \end{aligned}$$

Explicit Midpoint

- **Explicit midpoint** rule

$$x^{\left(k+\frac{1}{2},\star\right)} = x^{(k)} + f\left(x^{(k)},\ t^{(k)}\right)\frac{\Delta t}{2}$$
$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{2} f\left(x^{\left(k+\frac{1}{2},\star\right)},\ t^{(k)} + \frac{\Delta t}{2}\right).$$

- Explicit midpoint/trapezoidal are a representative of a powerful class of second-order methods called **predictor-corrector methods**: Euler (forward or backward) method is the predictor, and then (implicit or explicit) trapezoidal/midpoint method is the corrector.
- One can also consider these as examples of **multi-stage one-step** methods: the predictor is the first stage, the corrector the second.

Higher-Order Runge-Kutta Methods

- The idea in RK methods is to evaluate the function $f(x,t)$ several times and then take a time-step based on an average of the values.
- In practice, this is done by performing the calculation in **stages**: Calculate an intermediate approximation $x^\star$, evaluate $f(x^\star)$, and go to the next stage.
- The most celebrated Runge-Kutta methods is a **four-stage** fourth-order accurate RK4 method based on **Simpson's rule** for the integral:

$$x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left[x(s), s\right] ds$$

$$\approx x^{(k)} + \frac{\Delta t}{6}\left[f(x^{(k)}) + 4f(x^{(k+1/2)}) + f(x^{(k+1)})\right]$$

$$= x^{(k)} + \frac{\Delta t}{6}\left[f^{(k)} + 4f^{(k+1/2)} + f^{(k+1)}\right],$$

and we approximate $4f^{(k+1/2)} = 2f^{(k+1/2;1)} + 2f^{(k+1/2;2)}$.

RK4 Method

$$f^{(k)} = f\left(x^{(k)}\right), \quad x^{(k+1/2;1)}, = x^{(k)} + \frac{\Delta t}{2} f^{(k)}$$

$$f^{(k+1/2;1)} = f\left(x^{(k+1/2;1)},\ t^{(k)} + \Delta t/2\right)$$

$$x^{(k+1/2;2)} = x^{(k)} + \frac{\Delta t}{2} f^{(k+1/2;1)}$$

$$f^{(k+1/2;2)} = f\left(x^{(k+1/2;2)},\ t^{(k)} + \Delta t/2\right)$$

$$x^{(k+1;1)} = x^{(k)} + \Delta t\, f^{(k+1/2;2)}$$

$$f^{(k+1)} = f\left(x^{(k+1;1)},\ t^{(k)} + \Delta t\right)$$

$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{6}\left[f^{(k)} + 2f^{(k+1/2;1)} + 2f^{(k+1/2;2)} + f^{(k+1)}\right]$$

Intro to multistep Methods

$$x^{(k+1)} \approx x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left[x(s), s\right] ds.$$

- Euler's method was based on a piecewise constant approximation (extrapolation) of $f(s) \equiv f\left[x(s), s\right]$.
- If we instead integrate the linear extrapolation

$$f(s) \approx f\left(x^{(k)}, t^{(k)}\right) + \frac{f\left(x^{(k)}, t^{(k)}\right) - f\left(x^{(k-1)}, t^{(k-1)}\right)}{\Delta t}(s - t_k),$$

  we get the second-order **two-step Adams-Bashforth** method

$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{2}\left[3f\left(x^{(k)}, t^{(k)}\right) - f\left(x^{(k-1)}, t^{(k-1)}\right)\right].$$

- This is an example of a **multi-step method**, which requires keeping previous values of f.

In MATLAB

- In MATLAB, there are several functions whose names begin with

$$[\mathbf{t}, \mathbf{x}] = ode(f,\ [t_0, t_e],\ x_0,\ odeset(\dots)).$$

- *ode*23 is a second-order **adaptive explicit** Runge-Kutta method, while *ode*45 is a fourth-order version (try it first).
- *ode*23*tb* is a second-order **implicit** RK method.
- *ode*113 is a **variable-order explicit** multi-step method that can provide very high accuracy.
- *ode*15*s* is a **variable-order implicit** multi-step method.
- For implicit methods the Jacobian can be provided using the *odeset* routine – very important!

Rigid body motion

```
function dy = rigid(t,y)
dy = zeros(3,1);    % a column vector
dy(1) = y(2) * y(3);
dy(2) = -y(1) * y(3);
dy(3) = -0.51 * y(1) * y(2);
%-------------

opts = odeset('RelTol',1e-3,'AbsTol',[1e-4 1e-4 1e-5]);
[T,Y] = ode45(@rigid, [0 12], [0 1 1], opts);

plot(T,Y(:,1),'o-r', T,Y(:,2),'s-b', T,Y(:,3),'d-g');
xlabel('t'); ylabel('y'); title('RelTol=1e-3');
```

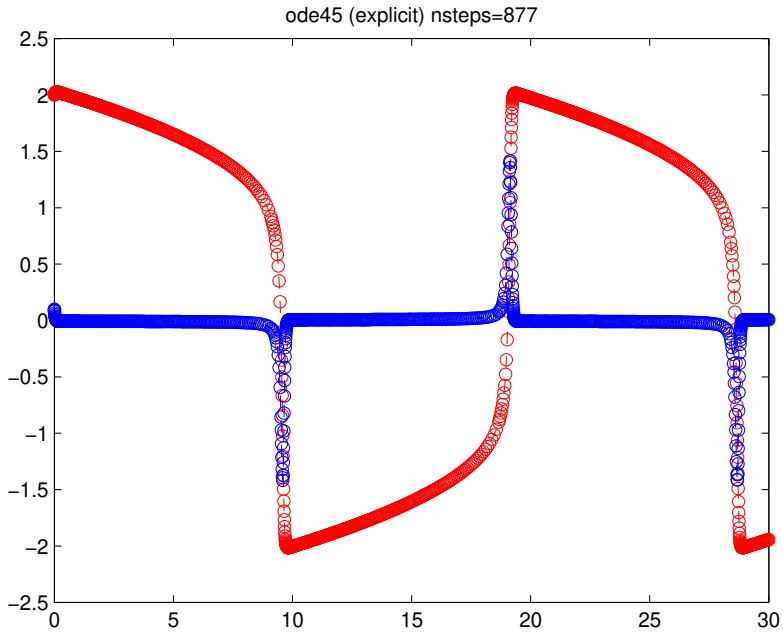
van der Pol equation

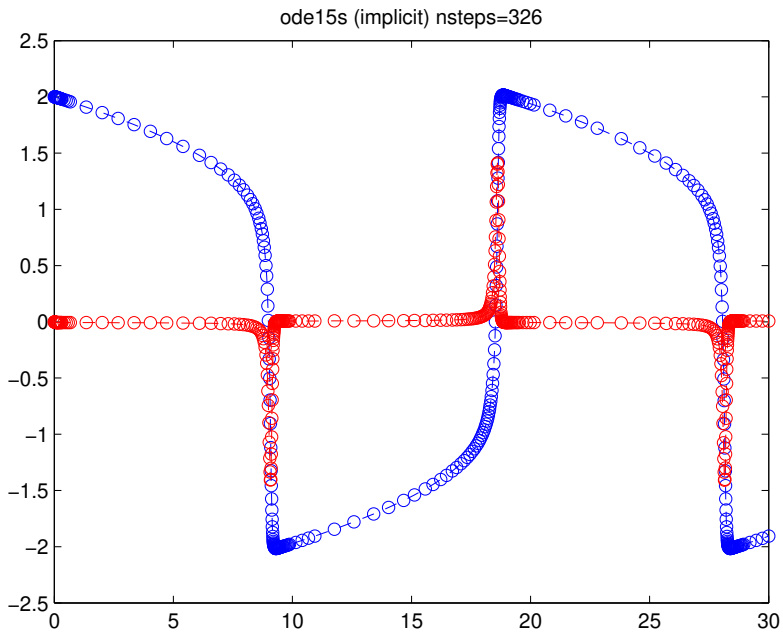
```
r=10; % Try r=100
f = @(t,y) [y(2); r*(1 - y(1)^2)*y(2) - y(1)];

figure(2); clf
[T,Y] = ode45(f,[0 3*r],[2 1]);
plot(T,Y(:,1),'o--r', T,Y(:,2)/r,'o--b')
title(['ode45 (explicit) nsteps=', int2str(size(T,1))]);

figure(3); clf
[T,Y] = ode15s(f,[0 3*r],[2 0]);
plot(T,Y(:,1),'o--b', T,Y(:,2)/r,'o--r')
title(['ode15s (implicit) nsteps=', int2str(size(T,1))]);
```

Stiff van der Pol system ($r = 10$)





A stiff problem is one where Δt has to be small even though the solution is smooth and a large Δt is OK for accuracy.

Zero Stability

- Consistency is not enough — we must also examine **error propagation** from one time step to another.
- A method is called **zero stable** if for all sufficiently small but finite Δt, introducing perturbations at each step (e.g., roundoff errors, errors in evaluating f) with magnitude less than some small ϵ perturbs the solution by at most $O(\epsilon)$.
- This simply means that errors do not increase but rather decrease from step to step.
- Denote our numerical approximation with time step size τ:

$$\mathbf{x}^{(k)} \approx \mathbf{x}\left(k\Delta t\right).$$

- A method is **convergent** if applying it to any system of ODEs where $\mathbf{f}$ is Lipshitz over a **finite time interval** $T > 0$ during which the ODE has a solution, the numerical approximation converges to that solution,

$$\lim_{\Delta t \to 0} \mathbf{x}^{(N=T/\Delta t)} = \mathbf{x}\left(T\right).$$

Convergence

- A central theorem in numerical methods for differential equations is that
  Any consistent method is convergent if and only if it is zero stable, or

$$\text{consistency} + \text{(zero) stability} = \text{convergence}.$$

- Note that we haven't given a precise definition to **zero stability** because in some sense it is defined as: the extra conditions that are needed to make a consistent method convergent.
- **Convergence is a statement about a limit, and does not imply a method will give reasonable answers for finite $\Delta t > 0$.**
  For that we will later introduce **absolute stability**.
- It can be shown that **one-step methods are zero-stable** if f is well-behaved (Lipschitz continuous w.r.t. second argument).

Stiff example

- In section 7.1 LeVeque discusses

$$x'(t) = \lambda (x - \cos t) - \sin t.$$

  with solution $x(t) = \cos t$ if $x(0) = 1$.
- If $\lambda = 0$ then this is very simple to solve using Euler's method, for example, $\Delta t = 10^{-3}$ up to time $T = 2$ gives error $\sim 10^{-3}$.
- For $\lambda = -10$, one gets an even smaller error with the same time step size.
- But for $\lambda = -2100$, results for $\Delta t > 2/2100 = 0.000954$ are completely useless: **method is unstable**.

Conditional Stability

- Consider the model problem for $\lambda < 0$:

$$x'(t) = \lambda x(t)$$
$$x(0) = 1,$$

with an exact solution that **decays exponentially**, $x(t) = e^{\lambda t}$.

- Applying Euler's method to this model equation gives:

$$x^{(k+1)} = x^{(k)} + \lambda x^{(k)} \Delta t = (1 + \lambda \Delta t)\, x^{(k)} \quad \Rightarrow$$

$$x^{(k)} = (1 + \lambda \Delta t)^k$$

- The numerical solution will **decay** if the time step satisfies the **stability criterion**

$$|1 + \lambda \Delta t| \leq 1 \quad \Rightarrow \quad \Delta t < -\frac{2}{\lambda}.$$

- Otherwise, the numerical solution will eventually blow up!

Unconditional Stability

- The above analysis shows that **forward Euler is conditionally stable**, meaning it is stable if $\Delta t < 2/|\lambda|$.
- Let us examine the stability for the model equation $x'(t) = \lambda x(t)$ for **backward Euler**:

$$x^{(k+1)} = x^{(k)} + \lambda x^{(k+1)} \Delta t \quad \Rightarrow \quad x^{(k+1)} = x^{(k)} / (1 - \lambda \Delta t)$$

$$x^{(k)} = x^{(0)} / (1 - \lambda \Delta t)^k$$

- We see that the implicit **backward Euler is unconditionally stable**, since for any time step

$$|1 - \lambda \Delta t| > 1.$$

Stiff Systems

- An ODE or a system of ODEs is called **stiff** if the solution evolves on widely-separated timescales and the fast time scale decays (dies out) quickly.
- We can make this precise for linear systems of ODEs, $\mathbf{x}(t) \in \mathbb{R}^n$:

$$\mathbf{x}'(t) = \mathbf{A}\left[\mathbf{x}(t)\right].$$

- Assume that $\mathbf{A}$ has an eigenvalue decomposition, with potentially **complex eigenvalues**:

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1},$$

and express $\mathbf{x}(t)$ in the basis formed by the eigenvectors $\mathbf{x}_i$:

$$\mathbf{y}(t) = \mathbf{X}^{-1}\left[\mathbf{x}(t)\right].$$

contd.

$$\mathbf{x}'(t) = \mathbf{A}\left[\mathbf{x}(t)\right] = \mathbf{X}\mathbf{\Lambda}\left[\mathbf{X}^{-1}\mathbf{x}(t)\right] = \mathbf{X}\mathbf{\Lambda}\left[\mathbf{y}(t)\right] \quad \Rightarrow$$

$$\mathbf{y}'(t) = \mathbf{\Lambda}\left[\mathbf{y}(t)\right]$$

- The different y variables are now **uncoupled**: each of the n ODEs is independent of the others:

$$y_i = y_i(0)e^{\lambda_i t}.$$

- Assume for now that all eigenvalues are **real and negative**, $\boldsymbol{\lambda} < 0$, so each component of the solution decays:

$$\mathbf{x}(t) = \sum_{i=1}^{n} y_i(0)e^{\lambda_i t}\mathbf{x}_i \quad \rightarrow \quad 0 \text{ as } t \rightarrow \infty.$$

Stiffness

- If we solve the original system using Euler's method, the time step must be smaller than the smallest stability limit,

$$\Delta t < \frac{2}{\max_i |\mathrm{Re}(\lambda_i)|}.$$

- A system is **stiff** if there is a strong **separation of time scales** in the eigenvalues:

$$r = \frac{\max_i |\mathrm{Re}(\lambda_i)|}{\min_i |\mathrm{Re}(\lambda_i)|} \gg 1.$$

- For non-linear problems $\mathbf{A}$ is replaced by the Jacobian $\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{f}(\mathbf{x}, t)$.
- In general, the Jacobian will have **complex eigenvalues** as well.

Absolute Stability

- We see now that for systems we need to allow λ to be a **complex number** but we can still look at scalar equations.
- A method is called **absolutely stable** if for $\mathrm{Re}(\lambda) < 0$ the numerical solution of the **scalar model equation**

$$x'(t) = \lambda x(t)$$

decays to zero, like the actual solution.

- We call the **region of absolute stability** the set of complex numbers

$$z = \lambda \Delta t$$

for which the numerical solution decays to zero.

- For systems of ODEs **all scaled eigenvalues of the Jacobian $\lambda_i \Delta t$ should be in the stability region.**

Stability regions

- For Euler's method, the stability condition is

$$|1 + \lambda \Delta t| = |1 + z| = |z - (-1)| \leq 1 \quad \Rightarrow$$

which means that z must be in a unit disk in the complex plane centered at $(-1, 0)$:

$$z \in \mathcal{C}_1(-1, 0).$$

- A general one-step method of order p applied to the **model equation** $x' = \lambda x$ where $\lambda \in \mathbb{C}$ gives:

$$x^{n+1} = R(z = \lambda \Delta t) x^n.$$

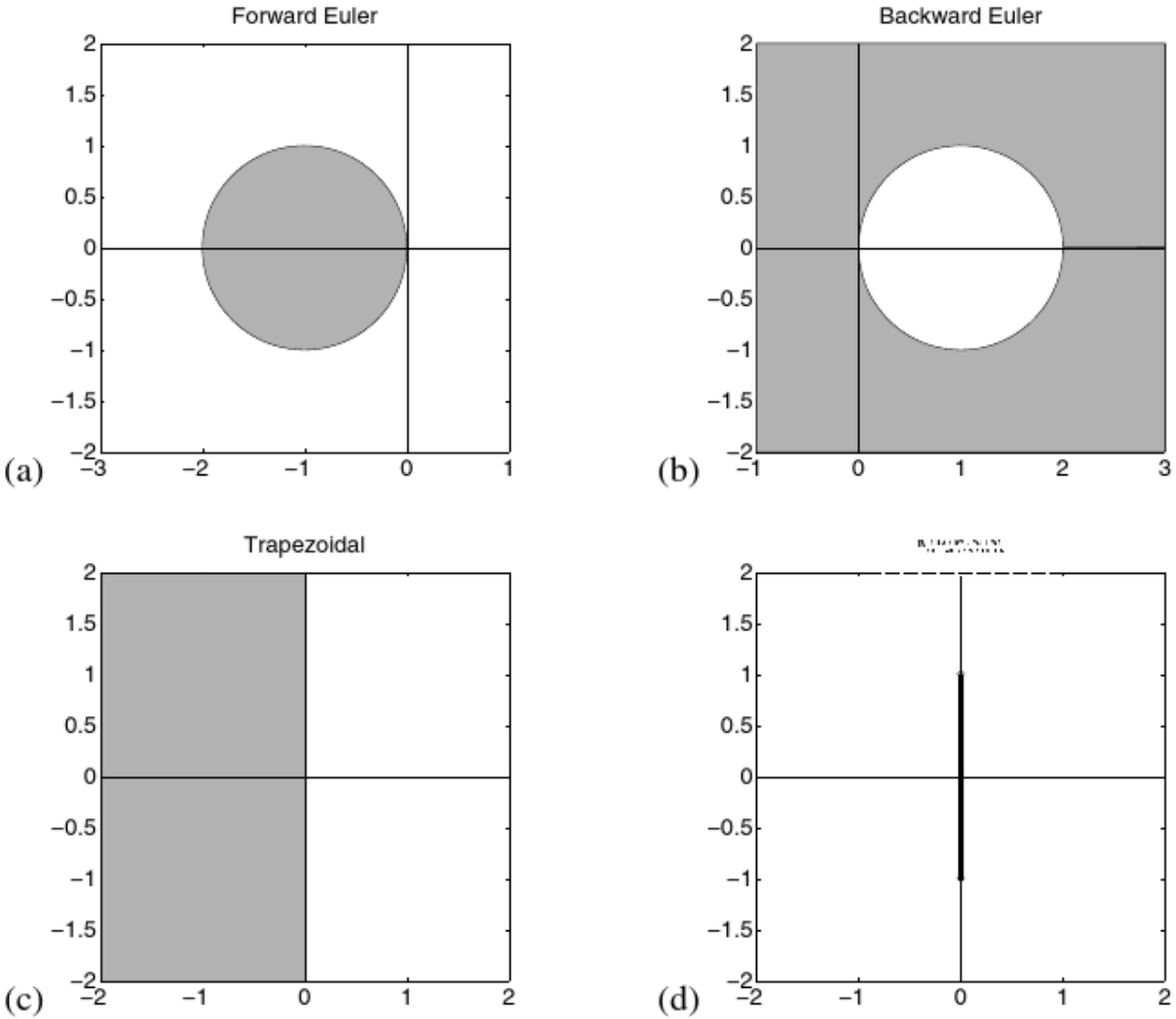
$$R(z) = e^z + O\left(z^{p+1}\right) \text{ for small } |z|.$$

- The **region of absolute stability** is the set

$$\mathcal{S} = \{z \in \mathbb{C} : |R(z)| \leq 1\}.$$

Simple Schemes

Forward/backward Euler, **implicit trapezoidal**, and leapfrog schemes



A-Stable methods

- A method is **A-stable** if its stability region contains the entire left half plane.
- The backward Euler and the implicit midpoint scheme are both A-stable, but they are also both implicit and thus expensive in practice!
- Theorem: **No explicit one-step method can be A-stable** (discuss in class why).
- Theorem: All explicit RK methods with r stages and of order r have the same stability region (discuss why).

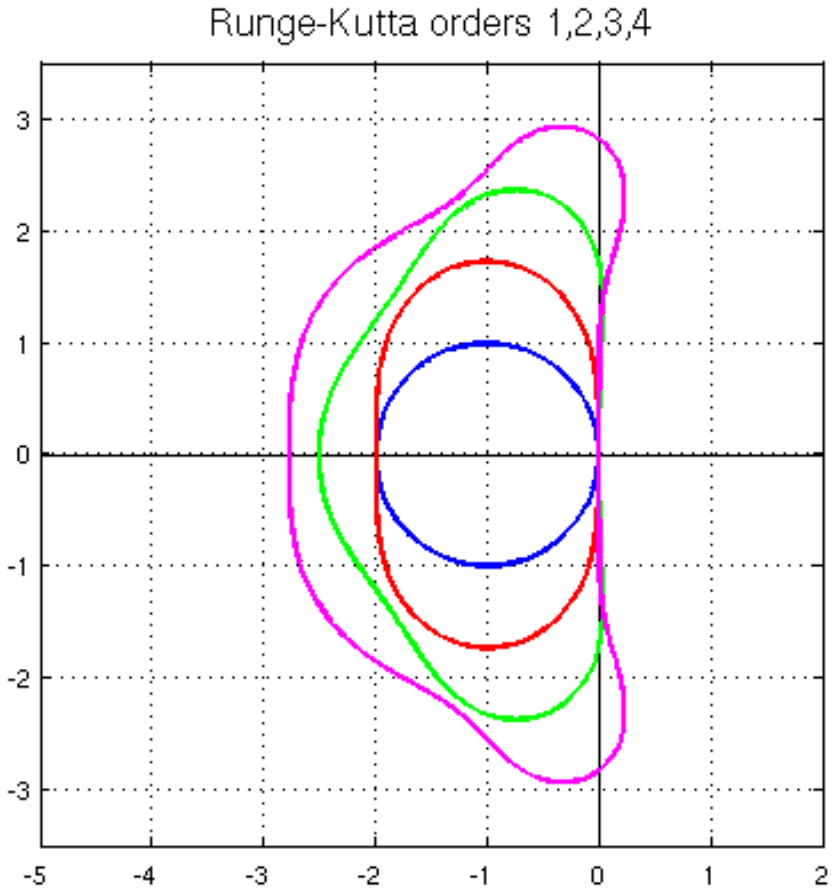
One-Step Methods

- Any r-stage **explicit** RK method will produce $R(z)$ that is a **polynomial** of degree r.
- Any r-stage **implicit** RK method has **rational** $R(z)$ (ratio of polynomials).
  The degree of the denominator cannot be larger than the **number of linear systems that are solved** per time step.
- RK methods give polynomial or rational approximations $R(z) \approx e^z$.
- A 4-stage explicit RK method therefore has

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

Explicit RK Methods

Stability regions for **all** r-stage explicit RK methods



One needs at least 3 stages to be stable for purely imaginary eigenvalues (hyperbolic PDEs later on).

Implicit Methods

- **Implicit methods are** generally **more stable** than explicit methods, and solving stiff problems generally requires using an implicit method.
- The price to pay is solving a system of non-linear equations at every time step (linear if the ODE is linear):
  This is best done using **Newton-Raphson**'s method, where the solution at the previous time step is used as an initial guess.
- For PDEs, the linear systems become large and implicit methods can become very expensive...

Implicit-Explicit Methods

- When solving PDEs, we will often be faced with problems of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t) = \text{stiff+non-stiff}$$

  where the stiffness comes only from $\mathbf{f}$.
- These problems are treated using **implicit-explicit (IMEX)** or **semi-implicit** schemes, which only treat $\mathbf{f}(\mathbf{x})$ implicitly (see HW4 for KdV equation).
- A very simple example of a second-order scheme is to treat $\mathbf{g}(\mathbf{x})$ using the **Adams-Bashforth** multistep method and treat $\mathbf{f}(\mathbf{x})$ using the implicit trapezoidal rule (**Crank-Nicolson** method), the **ABCN** scheme:

$$x^{(k+1)} = x^{(k)} + \frac{\Delta t}{2}\left[\mathbf{f}\left(x^{(k)},\ t^{(k)}\right) + f\left(x^{(k+1)},\ t^{(k+1)}\right)\right] + \Delta t\left[\frac{3}{2}g\left(x^{(k)},\ t^{(k)}\right) - \frac{1}{2}g\left(x^{(k-1)},\ t^{(k-1)}\right)\right].$$

Which Method is Best?

- As expected, there is no universally "best" method for integrating ordinary differential equations: It depends on the problem:
  - How stiff is your problem (may demand implicit method), and does this change with time?
  - How many variables are there, and how long do you need to integrate for?
  - How accurately do you need the solution, and how sensitive is the solution to perturbations (chaos).
  - How well-behaved or not is the function $f(x,t)$ (e.g., sharp jumps or discontinuities, large derivatives, etc.).
  - How costly is the function $f(x,t)$ and its derivatives (Jacobian) to evaluate.
  - Is this really ODEs or a something coming from a PDE integration (next lecture)?

Conclusions/Summary

- Time stepping methods for ODEs are **convergent if and only if they are consistent and stable**.
- We distinguish methods based on their **order of accuracy** and on whether they are **explicit** (forward Euler, Heun, RK4, Adams-Bashforth), or **implicit** (backward Euler, Crank-Nicolson), and whether they are **adaptive**.
- **Runge-Kutta methods** require more evaluations of f but are more robust, especially if adaptive (e.g., they can deal with sharp changes in f). Generally the recommended first-try (*ode*45 or *ode*23 in MATLAB).
- **Multi-step methods** offer high-order accuracy and require few evaluations of f per time step. They are not very robust however. Recommended for well-behaved non-stiff problems (*ode*113).
- For **stiff problems** an **implicit method** is necessary, and it requires solving (linear or nonlinear) systems of equations, which may be complicated (evaluating Jacobian matrices) or costly (*ode*15*s*).