PDE Spring 2016

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Review for final #2

Now we focus on second-order elliptic and parabolic PDEs
(Poisson, Laplace) (heat) also works for wave eq.

$$u_t = k\nabla^2 u + f, \quad k>0 \quad \text{— heat equation, parabolic}$$

$u(x, t=0) = \varphi(x)$ $\quad t>0$

+BCs

or

$$\nabla^2 u = f \text{ — Poisson}$$

$$\nabla^2 u = 0 \text{ — Laplace}$$

The easiest method of solution depends on the <u>boundary conditions</u>

1. <u>Infinite domain</u> (heat or wave equations), use Green's functions for parabolic.
2. <u>Finite domains</u> — Separation of variables

Diffusion equation on the whole real line or plane

2

$$\begin{cases} u_t = k u_{xx} & x \in \mathbb{R} \\ u(x,0) = \varphi(x) & IC \end{cases}$$

(no BCs!)

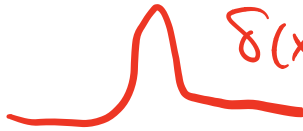
Solution obtained using Green's functions

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\varphi(y)\,dy$$

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-x^2/(4kt)}$$

Same works for wave equation also with a different G but because of the simple form of G it becomes identical to d'Alamber's formula.

$G_t = \nabla^2 G$
$G(x,0) = \delta(x)$ ← radially symmetric



$\delta(x,y) = \delta(\underline{r})$

<u>Import</u>: Easy to generalize to 2D/3D ③

In 2D: (separation of x and y)
$u = X(x,t)\,Y(y,t)$

$$G(x,y,t) = G(x,t)\,G(y,t)$$

(<u>recall midterm!</u>)

$$= \frac{1}{4\pi k t} e^{-(x^2+y^2)/(4kt)} \equiv$$

$$G(r,t) = \frac{1}{4\pi k t} e^{-r^2/(4kt)}$$

Formula is the same but now integral is over the whole plane:

$$u(x,y,t) = \int_{x'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty} G(x-x', y-y', t)\,\varphi(x',y')\,dx'\,dy'$$

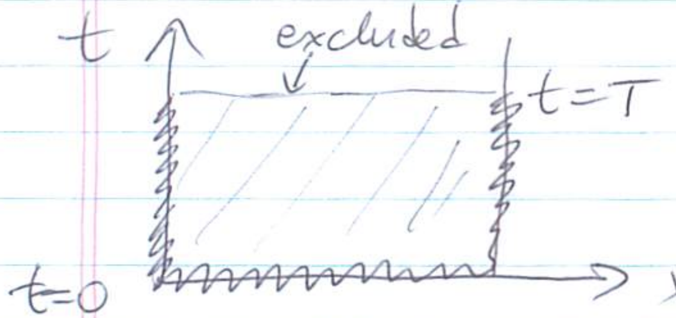
<u>Obvious</u> generalization to 3D/nD

Green's function difficult to figure out in general

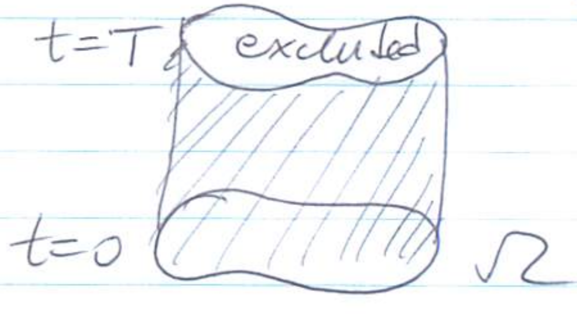
Maximum principle

For both the heat equation and the Laplace equation, an extremum principle applies.

For heat, the extremum is achieved on one of the boundaries of the space-time domain excluding $t = T$ boundary



1D



2D

For Laplace, the extremum is achieved on the boundary of the domain $\partial\Omega$

Bounded domains

General ∇^2 problem:

$$\begin{cases} u_t = \underset{\nabla^2}{\mathcal{L}} u + f(\vec{x}, t) \quad \text{for } \vec{x} \in \Omega \\ u(\vec{x}, t=0) = u_0(\vec{x}) \quad \text{IC} \\ u(\partial\Omega_1) = \varphi(\vec{x} \in \partial\Omega_1, t) \quad (\text{none for Laplace/Poisson}) \\ \frac{\partial u}{\partial \vec{n}} \cdot \vec{n}(\partial\Omega_2) = \psi(\vec{x} \in \partial\Omega_2, t) \end{cases}$$

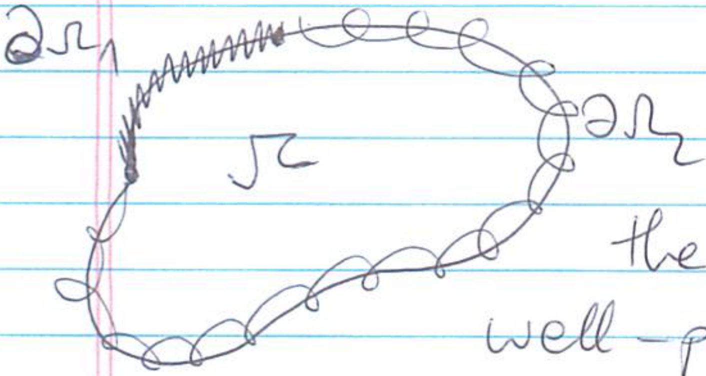
Dirichlet

Neumann

Neum: $\frac{\partial u}{\partial x} = \psi(y)$

$\frac{\partial u}{\partial y} = \psi(x)$

$$\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$$



<u>Note</u>

In order for the problem to be well-posed, depending on the equation and BCs, f and g <u>may</u> need to satisfy <u>additional cond.</u>

Use superposition to split into subproblems that may be easier to solve ⑥

$u = v + w$

P1 __Steady-state__ problem (elliptic) no time!

$$\begin{cases} \mathcal{L} v = 0 \\ + \text{ BCs for } v \text{ are the } \underline{\text{same}} \end{cases}$$

This handles the inhomogeneous BCs for us (so skip if homogeneous)

If φ and ψ do __not__ depend on t, then

$v \equiv v(\vec{x})$ only depends on $\vec{x}$

In general

$$v \equiv v(\vec{x}\,; t) \equiv v(\vec{x}, t)$$

depends on t, but t is only a __parameter__ for the solution since $\mathcal{L}$ only involves __spatial derivatives__

Note: In some problems, it may be simple to solve (7)

$$\mathcal{L}\,v = -f \quad \text{directly}$$

If so, this will speed up the process (recall problem in homework $u'' = Q$).

$$u_t = \mathcal{L}u + f \qquad u = v + w$$

$$w_t + v_t = \mathcal{L}w + \mathcal{L}v + f$$

P2

$$\begin{cases} w_t = \mathcal{L}w + f - v_t \\ w(\partial\Omega_1) = 0 \\ \dfrac{\partial w}{\partial \vec{n}} \cdot \vec{n}(\partial\Omega_2) = 0 \\ w(\vec{x}, t=0) = u_0(\vec{x}) - v(\vec{x}, 0) \end{cases}$$

homogeneous BCs

This problem we solve via the method of separation of variables, which here becomes the method of orthogonal series

$$\boxed{u = v + w}$$ Superposition

Note: If the original problem was not a heat equation but rather Poisson, the process would be the same

$$\begin{cases} \mathcal{L}u = f \\ + \text{BCs} \end{cases}$$

however, not necessary to split

P1: $\begin{cases} \mathcal{L}v = 0 \\ + \text{BCs} \end{cases}$ P2: $\begin{cases} \mathcal{L}w = f \\ + \underline{\text{homogeneous}} \text{ BCs} \end{cases}$

and again $u = v + w$

For final, do not try to memorize this. I will require you to show that indeed you have used the superposition correctly, i.e., to prove that $u = v + w$ solves the original BVP

Key idea: Handle inhomogeneous BCs first, then solve the rest with homogeneous BCs

Separation of variables

Consider first solving the Laplace equation with inhomogeneous BCs

$$\begin{cases} \mathcal{L}u = 0 \quad (\text{or } f) \\ u(\partial\Omega_1) = \varphi \\ \partial_n u(\partial\Omega_2) = \psi \end{cases}$$

If 1D it is easy to solve (ODE!)

This is solveable on a rectangular domain in 2D/3D using separation of variables

Assume separable solution

$$u = X(x)\, Y(y)$$

and plug into equation, then separate x and y pieces to get:

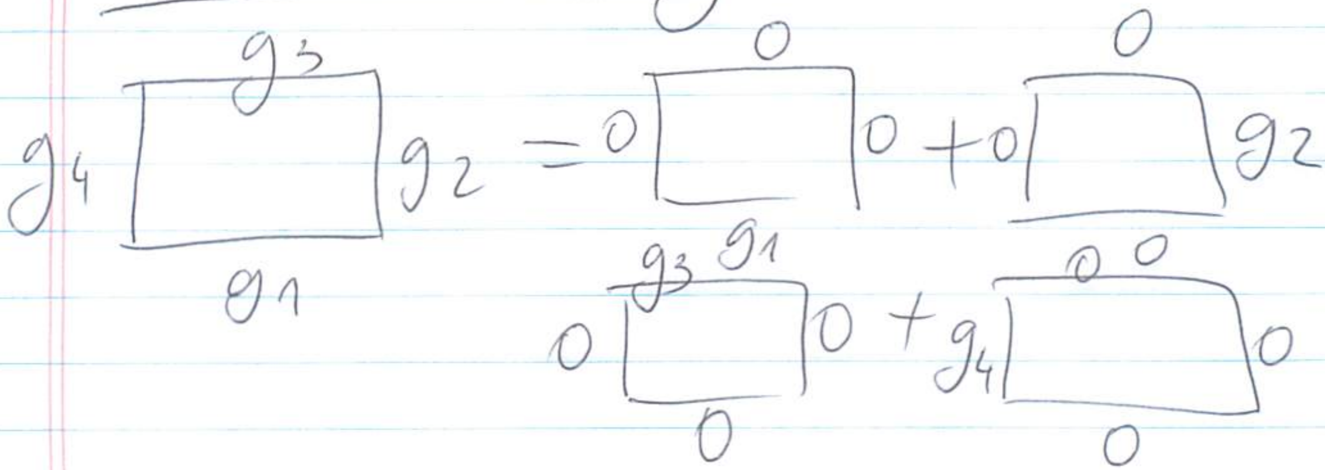
$$F(x) = G(y) = \text{const} = \lambda$$

and then solve ODEs

$$\begin{cases} F(x) = \lambda \\ G(y) = \lambda \end{cases}$$

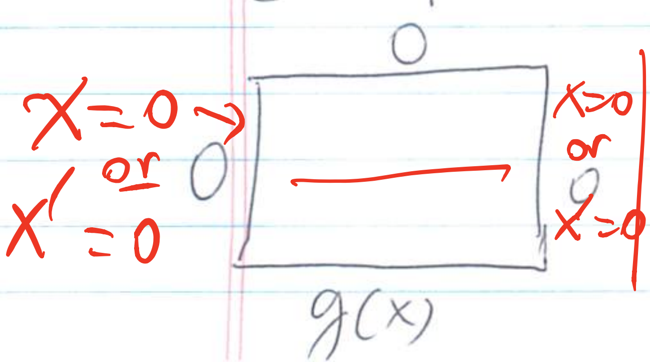
One by one, starting from the simplest one. To obtain possible values of λ, use BCs as follows.

First, split problem so that inhomogeneous BC is on only <u>one boundary</u>

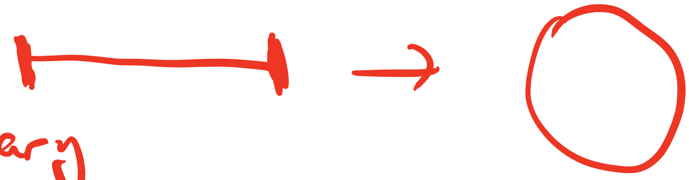


First, solve ode along homogeneous (or periodic) direction, e.g., for

$X'' = \lambda X$



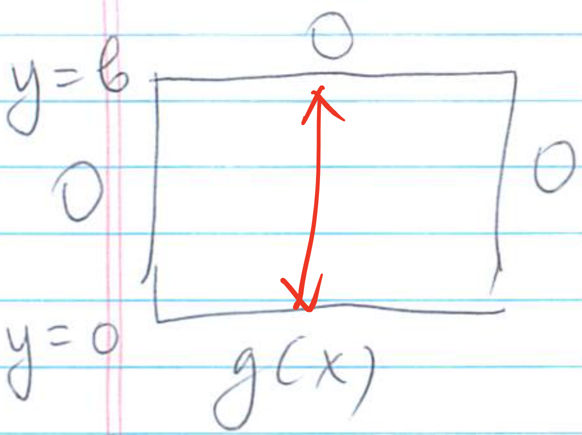
solve $F(x) = \lambda$ with homogeneous BCs first

Periodic:  topologically circle

no boundary

this will determine λ (along with its sign)

then, solve along the other direction, putting homogeneous BC along homogeneous boundaries, and 1 (unity) along others



Solve

$$G(y) = \lambda$$

$$Y(0) = 1, \quad Y(b) = 0$$

Once you get $X(x)$ and $Y(y)$, write solution as a superposition of separable solutions

✓ $u_t = \mathcal{L}u$ $\qquad u = \sum a_n X_n Y_n$

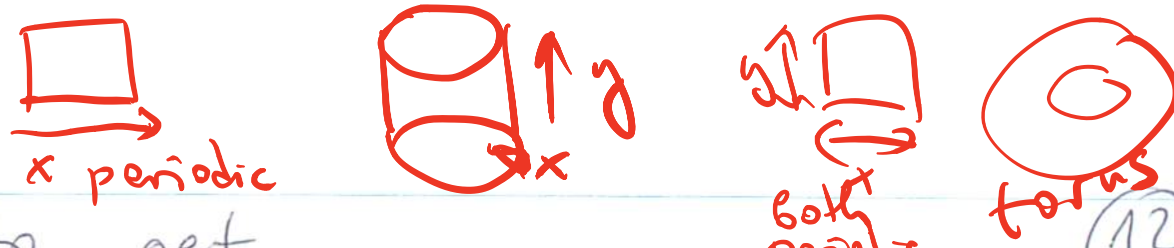
$$u = \sum_n a_n X_n(x) Y_n(y)$$

$$u_t = \sum a_n (X_n)_t Y_n + \sum a_n X_n (Y_n)_t \doteq \mathcal{L}u = \sum_n a_n X_n'' Y_n + \sum_n a_n X_n Y_n''$$

and plug into BC, e.g.

$$u(x, y=0) = g(x)$$

Periodic in 2D:



to get

$$\boxed{\sum a_n X_n(x) = g(x)}$$

Fourier series

which we need to solve for the coefficients a_n.

This is quite generally an orthogonal series, e.g., a Fourier series, since $X_n(x)$ were solutions of an equation of the form e.g. $X_n'' = -\lambda_n X_n$ + BCs.

$$\boxed{\mathcal{L} u = -\lambda u} + \text{homog. BCs}$$

where $\mathcal{L}$ was a self-adjoint operator. This implies that Eigenfunctions are orthogonal and eigenvalues are real

(L_2) Dot product (inner product):

$$(f, g) = \int_a^b f(x)\, \overline{g}(x)\, dx$$

or more generally

$$(f,g) = \int_{\Omega} f(\vec{x})\,\bar{g}(\vec{x})\,d\vec{x}$$

Ω ← over domain

If it is true that

$$(\bar{X}_n, \bar{X}_m) = 0 \text{ if } n \neq m$$

then the X_n's form an orthonormal basis and

$$\sum a_n X_n(x)$$

is an orthogonal series.

How to solve

$$\sum a_n X_n(x) = g(x) \quad \Big| \text{ (dot product) Multiply by } X_m(x)$$

$$\left(X_m, \sum_n a_n X_n\right) = (X_m, g) =$$

$$\sum_n a_n \underbrace{(X_m, X_n)}_{\neq 0 \text{ only for } m=n} = a_m (X_m, X_m)$$

$$\Rightarrow$$

$$\boxed{a_m = \frac{(X_m, g)}{(X_m, X_m)}} \checkmark$$

Examples

Fourier sine series

$$X'' = -\lambda X, \quad X(0) = X(a) = 0$$

Fourier cosine series

$$X'' = -\lambda X, \quad X'(0) = X'(a) = 0$$

Mixed series

$$X'' = -\lambda X, \quad X(0) = 0, \; X'(a) = 0$$

Periodic boundaries

(not really boundaries – domain is topologically a circle or a torus in 2D)

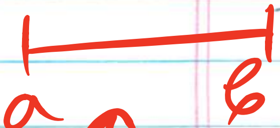
$$X'' = -\lambda X, \quad X(-\ell) = X(\ell)$$
$$X'(-\ell) = X'(\ell)$$

How to think about this geometrically?

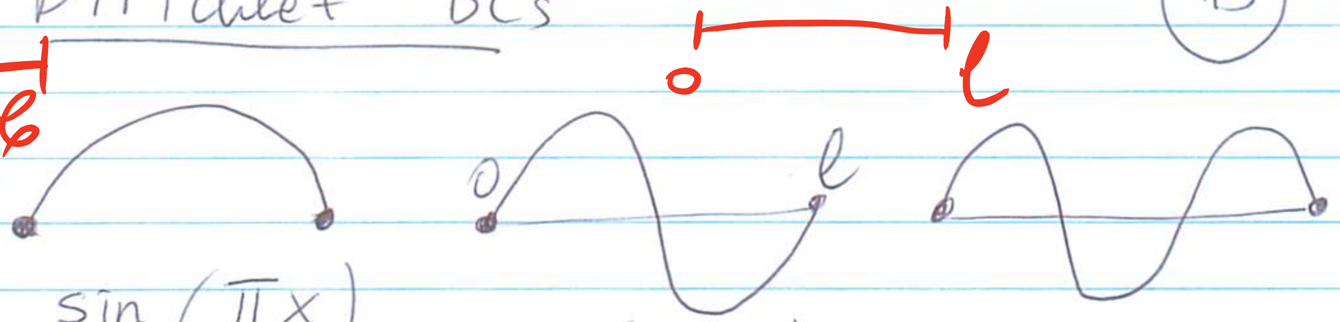
We know solutions are sine and cosine

Basis functions are trig functions $\sin(\underset{=}{\alpha} x + \underset{=}{\beta})$

Dirichlet BCs



Replace x by $(x-a)$ and $\ell = b-a$



$\sin\left(\frac{\pi x}{\ell}\right)$ $\quad \sin\left(\frac{2\pi x}{\ell}\right)$ $\quad \sin\left(\frac{3\pi x}{\ell}\right)$

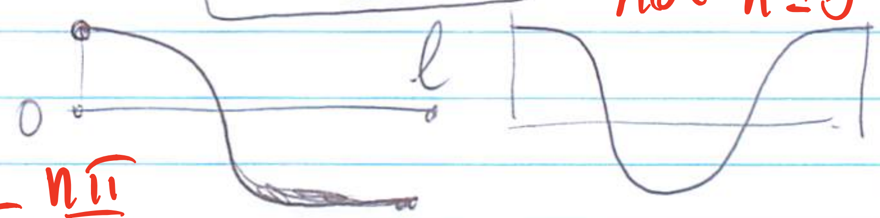
$\alpha_n = \frac{n\pi}{\ell}$

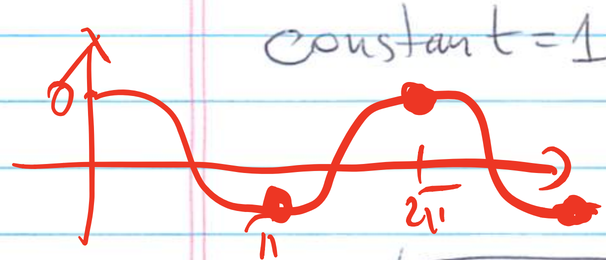
$\boxed{\sin\left(\frac{n\pi x}{\ell}\right)}$ $n = 1, 2, \ldots$ not $n=0$

Neumann BCs

$\cos(\alpha x)$

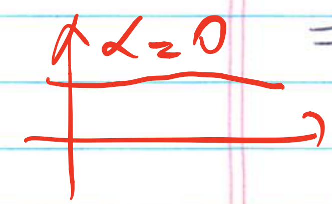
constant = 1





$\alpha = \frac{n\pi}{\ell}$

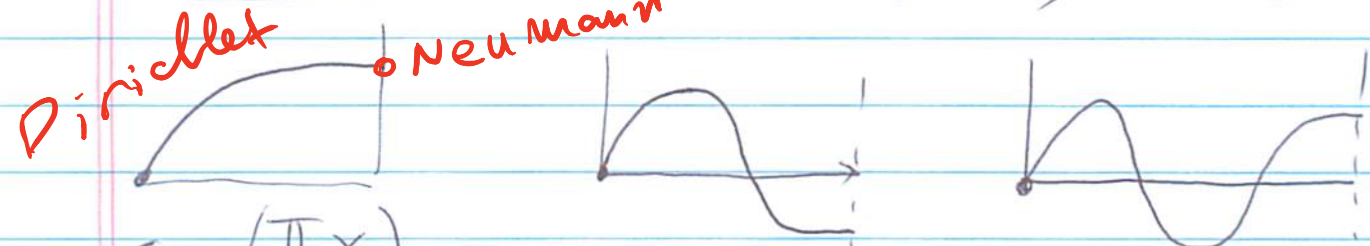
$\cos\left(\pi \frac{x}{\ell}\right)$ $\quad \cos\left(\frac{2\pi x}{\ell}\right)$



$\Rightarrow \boxed{\cos\left(\frac{n\pi x}{\ell}\right), \; n = 0, 1, 2, \ldots}$

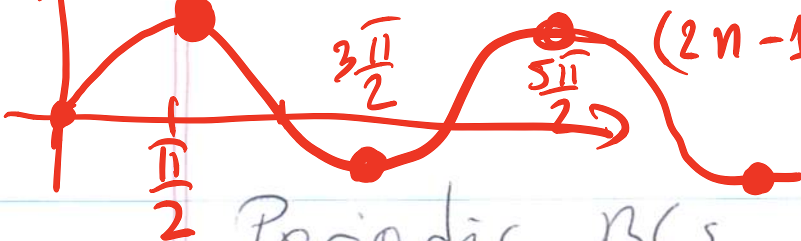
Mixed BCs

$X(0) = 0, \quad X'(\ell) = 0$



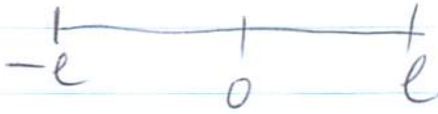
$\sin\left(\frac{\pi}{2}\frac{x}{\ell}\right)$ $\quad \sin\left(\frac{3\pi}{2}\frac{x}{\ell}\right)$ $\quad \sin\left(\frac{5\pi}{2}\frac{x}{\ell}\right)$

$\boxed{\sin\left((2n-1)\frac{\pi x}{2\ell}\right), \; n = 1, 2, \ldots}$



$(2n-1)\pi/2 = \alpha\ell$

constant



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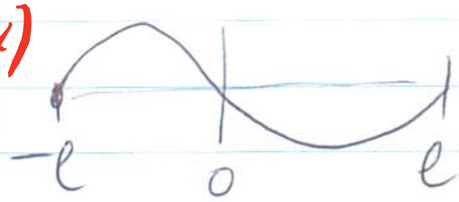
Periodic BCs

or 0 to $2\ell = L$

$\sin(\alpha x)$



$\cos(\alpha x)$

$\cancel{2}\ell\alpha = \cancel{2}\pi n$ $\quad \sin\left(\frac{n\pi x}{\ell}\right)$ $\qquad \cos\left(\frac{n\pi x}{\ell}\right)$

$\alpha = n\pi/\ell$

Any sine or cosine that has periodicity of 2ℓ works.
Best basis for calculations is complex one:

$$\boxed{\exp\left(i\frac{n\pi x}{\ell}\right), \; n = -\infty, \dots, 0, \dots +\infty}$$

In all cases the functions are orthogonal, so, for example

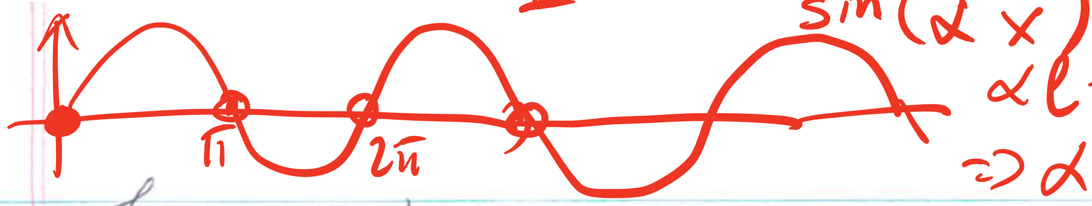
$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} = \varphi(x)$$

complex conjugate

$$\Rightarrow \quad c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \varphi(x) e^{-in\pi x/\ell} dx$$

denominator

Since $\int_{-\ell}^{\ell} e^{-in\pi x/\ell} e^{in\pi x/\ell} dx = 2\ell$!



$\alpha l = n\pi$
$\Rightarrow \alpha = n\pi / l$

The most general case of a 1D eigenvalue problem is the Sturm-Liouville:

$$\lambda u = \mathcal{L} u = -\left(p(x)\, u'(x) \right)' + q(x)\, u$$

$p(x) > 0$ and differentiable

$q(x) > 0$ and continuous

Eigenfunctions are orthogonal

Eigenvalues are positive

$0 < \lambda_1 < \lambda_2 \ldots$

This means we can expand functions (e.g., unknown solution) as a series of such orthogonal functions.

Practice: $u'' \quad u = -\lambda u$

$u(0) = u(1) = 0$

or even $\mathcal{L} = 3u'' - \underline{2u'} + u$

$u_t = k u_{xx} + \underline{ru} \iff w = e^{-rt} u$

$w_t = k w_{xx}$ or keep ru

Basic idea in <u>all</u> cases: (18)

Expand solution in series of eigenfunctions.

Find eigenfunctions in 2D by using separation of variables

$$u = X(x)\, Y(y) \qquad \checkmark\, X'' = -\lambda X, \quad Y'' = (\lambda - \lambda_x) Y$$

to convert to two one-dimensional eigenvalue problems, which are easy to solve as ODEs.

$\rightarrow \mathcal{L} u = \lambda u$ in 2D

$$\lambda_{n,m} = \lambda_x + \lambda_y = \left(\frac{n\pi}{\ell_x}\right)^2 + \left(\frac{m\pi}{\ell_y}\right)^2$$

$\sum_{n,m}$

<u>Example</u>

Heat equation with source and initial condition

$$(P2)\begin{cases} u_t = \mathcal{L} u + f \\ u(\partial\Omega) = 0 \\ u(t=0) = u_0 \end{cases}$$

<u>Step 1</u>: Find eigenfunctions of $\mathcal{L}$

$\mathcal{L} u_n = \lambda_n u_n$ + BCs

ODE for ?

$$u(\vec{x}, t) = \sum_n a_n(t)\, u_n(\vec{x})$$

If Poisson no time dependence

$$f(\vec{x},t) = \sum f_n(t)\, u_n(\vec{x})$$ 19

$$\Rightarrow \quad f_n(t) = \frac{\int_\Omega f(\vec{x},t)\, u_n(\vec{x})\, d\vec{x}}{\int_\Omega |u_n(\vec{x})|^2\, d\vec{x}}$$

Since u_n are orthogonal eigenfunctions

$$\boxed{\mathcal{L} u_n = \lambda_n u_n}$$

Plug back into PDE $u_t = \mathcal{L}u + f$

$$u_t = \sum a_n' \, u_n$$

$$\mathcal{L} u = \sum a_n (\mathcal{L} u_n) = \sum a_n \lambda_n u_n$$

$$\Rightarrow \sum a_n' u_n = \sum a_n \lambda_n u_n + \sum f_n u_n$$

$\Rightarrow$ (since u_n linearly independent)

$$\boxed{a_n' = \lambda_n a_n + f_n(t)}$$ ODE to solve

If Poisson $0 = \lambda_n a_n + f_n \Rightarrow a_n = -f_n/\lambda_n$

This ODE can be solved using Duhamel's principle

First solve $a_n' = \lambda_n a_n \Rightarrow$

$$a_n(t) = a_n(0)\, e^{\lambda_n t}$$

and thus

$$a_n(t) = a_n(0)\, e^{\lambda_n t} + \int_0^t f_n(\tau)\, e^{\lambda_n (t-\tau)}\, d\tau$$

How to determine $a_n(0)$?

Go back to initial conditions

$$u_n(x, t=0) = \sum a_n(0)\, u_n(x) = u_0(x)$$

$$\Rightarrow a_n(0) = \frac{\int_\Omega u_0 u_n \, d\vec{x}}{\int_\Omega (u_n)^2 \, d\vec{x}}$$

Same ideas just recycled over and over again!

Example

$$\begin{cases} u_t = \mathcal{L}u + f \\ u(\partial\Omega, t) = f(x,t) \\ u(x, t=0) = u_0(x) \end{cases}$$

① separ. of vars
② superposition
③ Fourier series

superposition

$$u_t = \underbrace{2(u_{xx} + u_{yy}) - u}_{\mathcal{L}u} + \overbrace{\sin(5\pi x / l_x)}^{f(x)}$$

$$\begin{cases} u(x=0, y, t) = u(x=l, y, t) = 0 \\ u_y(x, y=0, t) = 0 \\ u(x, y=l_y, t) = xe^t \end{cases}$$

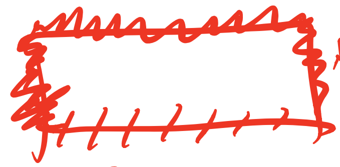
or ~~try~~ y is periodic

$$u(x,y,t=0) = \sin\left(\frac{15\pi x}{l_x}\right) y$$

Step 1:

$$\begin{cases} \mathcal{L}u = 2(u_{xx} + u_{yy}) - u = \lambda u \\ + \text{BCs} \end{cases}$$

homogeneous
+ BCs



$\frac{\partial u}{\partial y} = 0$

& $\mathcal{L} u_n = \lambda u_n$ + hom. BCs
not f, no inhom. BCs

$$u_n(x,y) = X(x)\,Y(y)$$

$$\frac{2X''}{X} - \lambda = \mu$$

$$-2Y'' - Y = \lambda Y \quad \checkmark$$

$$2(u_{xx} + u_{yy}) - u = \lambda u$$

$$\frac{2X''Y + 2XY'' - XY}{XY} = \frac{\lambda XY}{XY}$$

$$2\frac{X''}{X} + 2\frac{Y''}{Y} - 1 = \lambda$$

$$f(x) = g(y) = \mu$$

$$2\frac{X''}{X} - \lambda = -2\frac{Y''}{Y} - 1 = \mu$$

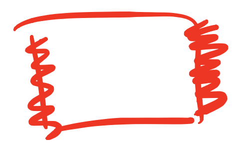
$$\begin{cases} 2X'' - \lambda X = \mu X & (1) \\ 2Y'' - Y = \mu Y & (2) \end{cases}$$

$$\|$$

$$\begin{cases} X'' = \left(\frac{\lambda + \mu}{2}\right) X & (1) \\ Y'' = -\left(\frac{\mu + 1}{2}\right) Y & (2) \end{cases}$$

$$\begin{cases} X'' = \tilde{\lambda} X \\ Y'' = \tilde{\tilde{\lambda}} Y \end{cases}$$

we've already seen

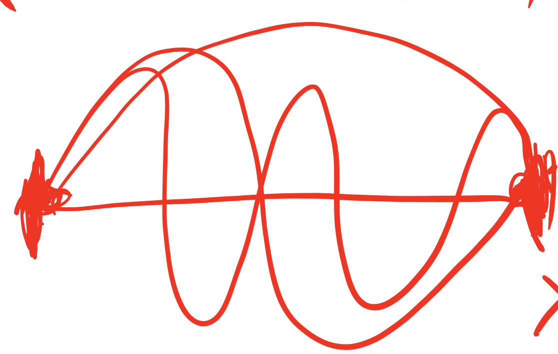


BCs:

$u(0,y) = u(l_x, y) = 0$

$\Downarrow$ $u = XY$

$$\begin{cases} X(0) = X(l_x) = 0 \\ X'' = \tilde{\lambda} X \end{cases}$$



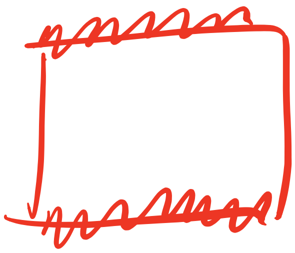
$X = \sin\left(\underset{=}{n}\pi \frac{x}{l_x}\right)$

$X'' = -\left(\frac{n\pi}{l_x}\right)^2 \sin\left(\frac{n\pi x}{l_x}\right)$

$=$

$$X'' = \tilde{\lambda} X$$

$$\boxed{\tilde{\lambda} = -\left(\frac{n\pi}{l_x}\right)^2 = \frac{\lambda + \mu}{2}}$$

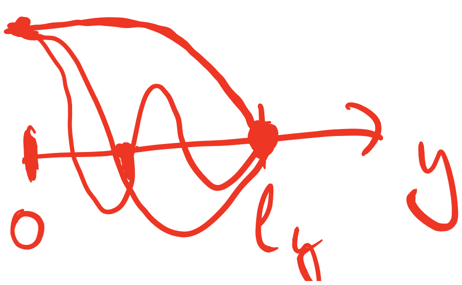


$$\begin{cases} u(x, l_y) = 0 & (\text{Dirichlet}) \\ u_y(x, 0) = 0 \end{cases}$$

$$u = XY$$

$$\begin{cases} Y(l_y) = 0 \\ Y'(0) = 0 \end{cases}$$

$$Y'' = \tilde{\lambda} Y$$



$$Y = \cos\left(\frac{(2n+1)\pi}{2}\,\frac{y}{l_y}\right)$$

$$n = 0, 1, 2, \ldots, \infty$$

$$(2n+1)\frac{\pi}{2} = \alpha l_y$$

$$\alpha = \frac{(2n+1)\pi}{2 l_y}$$

$$Y = \cos\left(\frac{2n+1}{2}\frac{\pi y}{l_y}\right)$$

$$Y'' = -\left(\frac{2n+1}{2}\frac{\pi}{l_y}\right)^2 Y$$

$$\boxed{\tilde{\lambda} = -\left(\frac{2n+1}{2}\frac{\pi}{l_y}\right)^2 = -\frac{\mu+1}{2}}$$

$$
\begin{cases}
\mu = \dots \\
\lambda = \dots \\
\quad = \\
u_{n,m} = X_n Y_m = \sin\left(\dfrac{n\pi x}{\ell_x}\right) \cos\left(\dfrac{2m+1}{2}\pi \dfrac{y}{\ell_y}\right)
\end{cases}
$$

Superposition

P1 + P2

inhomog. BCs. everything else

$u = v + w$

v: P1, w: P2

particular Solution

P1:

$$\mathcal{L}v = 2(\partial_{xx} + \partial_{yy})v - v = \underline{\underline{0}}$$

No v_t - Poisson eq.

$$v(0, y) = v(l_x, y) = 0$$

not

$$v_y(x, 0) = 0$$

$$v(x, l_y) = xe^t$$

$$v = X(x)Y(y)$$

P2 Everything that remains

$$\left.\begin{aligned} &u_t = \Delta u + f \\ &\text{inh BCs} \\ &u(t=0) = u_0 \end{aligned}\right\} \Rightarrow \begin{matrix} P1 \\ + \\ P2 \end{matrix} \qquad u = v + w$$

$$\text{P1}: \begin{cases} 0 = \Delta v \\ \text{inh BCs} \leftarrow \end{cases}$$

$+$

$$\text{P2}: \begin{cases} w_t = \Delta w + f - v_t \\ \text{hom BCs} \leftarrow \\ w(t=0) = u_0 - v_{(t=0)} \end{cases}$$

P

Eigenfunctions of Δ

$$w(x,y,t) = \sum_{n,m} a_{nm}(t)\, X_n(x)\, Y_m(y)$$

sin

cos

ODE for a_{nm}

$$w_t = \nabla^2 w + f - \varphi_t$$

$$\sum_{n,m} a'_{nm}(t) \cancel{X_n(x)} \cancel{Y_m(y)} =$$

$$\underbrace{= \sum_{n,m} \lambda_{n,m} \cancel{X_n(x)} \cancel{Y_m(y)} + \sin\left(\frac{5\pi x}{\ell_x}\right) - \varphi_t(x,y)}_{\displaystyle \| \quad \sum_{n,m} \underline{b_{n,m}(t)} \cancel{X_n(x)} \cancel{Y_m(y)}}$$

$$\begin{cases} a'_{nm} = +\lambda_{n,m} a_{nm} + b_{n,m}(t) \\ a_{nm}(t=0) = \overset{c_{nm}}{\text{initial condition of PDE}} \end{cases}$$

$$u(x, y, t=0) = \sin\left(\frac{15\pi x}{\ell_x}\right) y$$

$$= \sum_{n,m} c_{n,m} X_n(x) Y_m(y)$$

$$X_n(x) Y_m(y) = \sin\left(\frac{n\pi x}{\ell_x}\right) \cos\left(\frac{2m+1}{2} \frac{\pi y}{\ell_y}\right)$$

$$c_{n,m} = ?$$

$c_{n,m} \neq 0$ only if $n = 15$

$$f(y) = y = \sum c_{15,m} \cos\left(\frac{2m+1}{2} \frac{y}{\ell_y}\right)$$

$$c_{15,m} = \frac{\int y \cos\left(\frac{2m+1}{2} \frac{y}{\ell_y}\right) dy}{\int \cos^2\left(\frac{2m+1}{2} \frac{y}{\ell_y}\right) dy}$$