PDE Spring 2016

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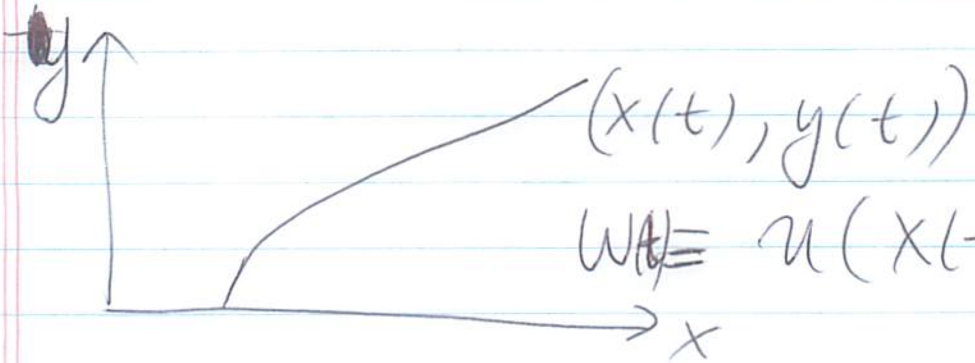
Review for final exam

First-Order PDEs

1 Method of Characteristics



$$w(t) = u(x(t), y(t))$$

Find curves $(x(t), y(t))$ such that from

$$\boxed{p u_x + q u_y = f} = \frac{dw}{dt}$$

$$\frac{dw}{dt} = \frac{dw(x(t), y(t))}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt}$$

$$= u_x\, p + u_y \cdot q = f$$

$$\boxed{p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}, \quad \frac{dw}{dt} = f}$$

Initial conditions get propagated along characteristics.

2 Change of coordinates

Transform into new coordinates to get a PDE of the form

$$u_{x'} = g \qquad \text{(only \underline{one} derivative)}$$

This is best when p and q are constant, since the new coordinate system is just a rotation

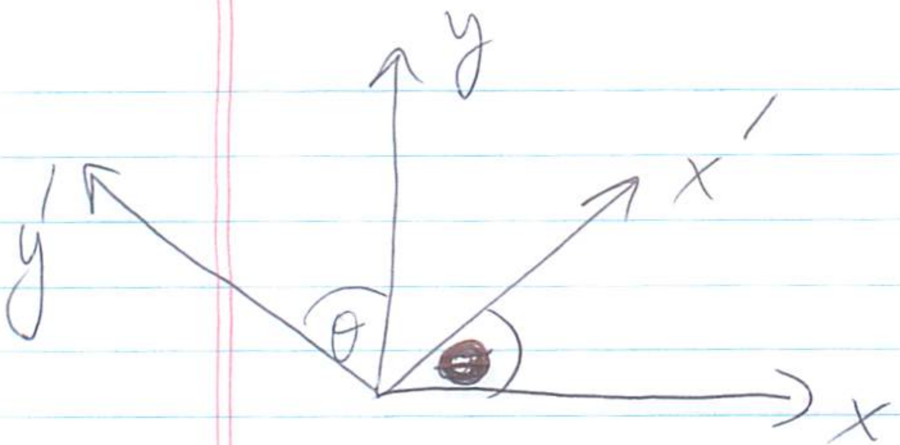
For

$$a u_x + b u_y = \text{something}$$

Introduce new coordinates

$$\begin{cases} x' = \alpha x + \beta y \\ y' = \beta x - \alpha y \quad \text{or } -\beta x + \alpha y \end{cases}$$

Why? We want $x' \perp y' \Rightarrow$



$\alpha = \cos\theta$

$\beta = \sin\theta$

$(\alpha, \beta) \perp (-\beta, \alpha)$ since dot product is zero

$(\alpha, \beta) \perp (\beta, -\alpha)$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}$$

$$\Rightarrow a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} =$$

$$\left(\frac{\partial x'}{\partial x}\right) a\, u_{x'} + \left(\frac{\partial y'}{\partial x}\right) a\, u_{y'}$$

$$\left(\frac{\partial x'}{\partial y}\right) b\, u_{x'} + \left(\frac{\partial y'}{\partial y}\right) b\, u_{y'}$$

$$= (a\alpha + \beta b)\, u_{x'} + (a\beta - b\alpha)\, u_{y'}$$

We want to have $(a\beta - b\alpha) = 0$

Choose $\alpha = a$, $\beta = b$

$\Rightarrow$

$$a u_x + b u_y = (a^2 + b^2) u_{x'}$$

So now we get ODE

$$(a^2 + b^2) u_{x'} = f$$

Example for practice (in recitation)

Method of characteristics:

P1 Solve

$$y u_x + x u_y = 0 \text{ with}$$

$$\text{IC} \quad u(0, y) = y$$

In which region of the plane is the solution uniquely determ?

P2 Method of change of coordinates

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

PDE classification
for second-order PDEs

$$\mathcal{L}u = a u_{xx} + 2b u_{xy} + c u_{yy} = 0$$

$$d = b^2 - ac$$

The discriminant determines

1 $\begin{cases} d > 0 \text{ hyperbolic } \Rightarrow \\ \mathcal{L} = (\alpha \partial_x + \beta \partial_y)(\gamma \partial_x + \delta \partial_y) \\ \Rightarrow \text{Change coordinates so that} \\ \mathcal{L} = \partial_s \partial_t = \partial_{st} \end{cases}$

2 $\begin{cases} d = 0 \quad \text{parabolic} \\ \mathcal{L} = (\alpha \partial_x + \beta \partial_y)^2 \\ \mathcal{L} = \partial_{ss} \end{cases}$

3 $\begin{cases} d < 0 \\ \mathcal{L} \text{ cannot be factored in real factors} \\ \mathcal{L} = \partial_{ss} + \partial_{tt} \end{cases}$

The Wave Equation
or general hyperbolic eqs

$$\mathcal{L} = a u_{xx} + 2b u_{xy} + c u_{yy}$$
$$= (\alpha \partial_x + \beta \partial_y)(\gamma \partial_x + \delta \partial_y)$$
$$= \underbrace{\qquad}_{\partial_s} \quad \underbrace{\qquad}_{\partial_t}$$

Find linear transformation

$$\begin{cases} x = \alpha s + \gamma t \\ y = \beta s + \delta t \end{cases}$$

in order to get

$$\begin{cases} \partial_s = \alpha \partial_x + \beta \partial_y \\ \partial_t = \gamma \partial_x + \delta \partial_y \end{cases}$$

then solve

$$\partial_{st} u = f(s,t)$$

$$\partial_s (\partial_t u) = f \quad \text{first}$$

$$\partial_t u = \int^s f \, ds' + A(t)$$

$$u = \int^t \int^s f \, ds' dt' + B(t) + C(s)$$

Then we use $\left\{\begin{matrix} ICs \\ BCs \end{matrix}\right\}$ to figure out what $B(t)$ and $C(s)$ are.

Examples for practice

P1 : 4.8 in APDE

$$u_{xx} - 2u_{xy} - 3u_{yy} = 0$$

(is it hyperbolic?)

$$\begin{cases} u(x,0) = g_0(x) \\ u_t(x,0) = g_1(x) \end{cases}$$

P2 : 4.9 in APDE

Done in recitation 2/19 and notes on homepage

Hint for P1 :

Answer :

$$u(x,y) = \frac{1}{4}\left[3g_0\left(x + \frac{y}{3}\right) + g_0(x-y)\right] + \frac{3}{4}\int_{x-y}^{x+y/3} g_1(s)\,ds$$

$$F'(x) = f(x) \Rightarrow F(x) = \int^x f(s)\,ds = \int_a^x f(s)\,ds + C$$

Solving ODEs

(1) Separable $\quad \frac{dy}{dx} = \left(f(x)\right)^{-1} g(y)$

$$\frac{dx}{f(x)} = \frac{dy}{g(y)} \Rightarrow \int_x \frac{dx}{f(x)} = \int_y \frac{dy}{g(y)}$$

(don't forget integration constants)

(2) First-order linear

$$y' + p(x)y = q(x)$$

Multiply by integrating factor

$$P(x) = \exp\left(\int^x p(s)\,ds\right)$$

to convert to

$$\frac{d}{dx}\big(\overbrace{P(x)\,y}^{F(x)}\big) = \overbrace{P(x)\,q(x)}^{f(x)}$$

and now integrate this

Practice: $y' + 2xy = \sqrt{x}$, $y(0) = 3$

Second - Order

Constant - coefficient linear:

$$ay'' + by' + cy = 0$$ (what if non-zero?)

$$y = c_1 y_1(x) + c_2 y_2(x)$$

Look for solutions

$$y = e^{mx}$$

and plug into equation (don't try to remember this!)

$$am^2 + bm + c = 0$$

If roots are unequal

$$y_1 = e^{m_1 x} \qquad y_2 = e^{m_2 x}$$

If real and equal $m_1 = m_2 = m$

$$y_1 = e^{mx} \qquad y_2 = x e^{mx}$$

If complex conjugate $\alpha \pm i\beta$

$$y_1 = e^{\alpha x} \sin \beta x \qquad y_2 = e^{\alpha x} \cos \beta x$$

Special "easy" case

$$y'' = -\lambda y$$

Solutions are sums of exponentials or trig functions depending on sign:

$$y'' = -a^2 y \Rightarrow y_1 = \cos ax$$
$$y_2 = \sin ax$$

$$y'' = a^2 y \Rightarrow y_1 = e^{-ax}$$
$$y_2 = e^{+ax}$$

Special equation

Cauchy-Euler

$$ax^2 y'' + bxy' + cy = 0$$

has solutions of power-law form

$$y = x^m$$

If equal roots x^m and $x^m \ln x$