February 7th, 2020

Topics: verifying solution to a PDE, principle of continuum superposition and general solution to IVP for heat equation, general solution via integration, well-posedness.

1. Verify that the function
   \[ G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \]  
   satisfies the heat equation \( u_t = ku_{xx}, \) for \( t > 0. \)

2. Verify that the general solution to the following IVP (Initial Value Problem)
   \[
   \begin{cases}
   u_t = ku_{xx}, & \text{for } -\infty < x < +\infty, \ t > 0 \\
   u(x, t = 0) = \phi(x)
   \end{cases}
   \]
   is given by
   \[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy. \]  
   (For \( t > 0 \), use Problem (1); we also showed how this is an example of the principle of continuum superposition).

3. Solve the IVP in Problem (2) with initial condition \( \phi(x) = e^{-x}. \) In particular, use the general formula (2) to show that the solution is given by
   \[ u(x, t) = e^{kt-x}. \]  
   (In computing the integral, use the identity \( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1 \)).

4. Find the general solution to the equation
   \[ u_{xt} + 3u_x = 1 \]  
   (we used the substitution \( v = u_x \)).

5. Consider the following Cauchy problems for Laplace’s equation:
   \[
   \begin{cases}
   v_{xx} + v_{yy} = 0, & \text{for } -\infty < x < +\infty, \ y > 0 \\
   v(x, 0) = 0 \\
   v_y(x, 0) = 0
   \end{cases}
   \]  
   \[
   \begin{cases}
   u_{xx} + u_{yy} = 0, & \text{for } -\infty < x < +\infty, \ y > 0 \\
   u(x, 0) = 0 \\
   u_y(x, 0) = e^{-\sqrt{n}} \sin(nx)
   \end{cases}
   \]  
   Show that \( (ii) \) is an ill-posed problem; in particular, show that it is not stable with respect to boundary data. (use that \( v(x, y) = 0 \) solves \( (i) \) and \( u(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \sinh(ny) \) is a solution to \( (ii) \); look at what happens for large \( n \).)
February 14th, 2020

Topics: ODEs review, d’Alembert’s formula, differential operators in polar coordinates, method of characteristics, method of coordinates, ill-posedness of backwards heat equation.

1. ODEs Review:
   • 1st order equations, separable equations: \( \frac{dy}{dx} = f(x)g(y) \). For example, solve IVP
     \[
     \begin{cases}
     y' = e^{-y}(2x - 4), \\
y(5) = 0
     \end{cases}
     \]
   • 1st order linear equations, solvable by integrating factor method: \( \frac{dy}{dx} + p(x)y = q(x) \). For example, solve IVP
     \[
     \begin{cases}
     y' = 5x - \frac{3y}{x}, \\
y(1) = 2
     \end{cases}
     \]
   • 2nd order equations with constant coefficients: \( ay'' + by' + cy = d(x) \). For example, solve BVP
     \[
     \begin{cases}
     u'' - u = x, \quad \text{for } 0 < x < 2\pi \\
u(0) = u(2\pi) = 0
     \end{cases}
     \]
     Also find the general solutions to two important examples: (i) \( u'' + a^2u = 0 \), and (ii) \( u'' - a^2u = 0 \) (\( a \) is constant).

2. (Homework 1, Problem 3) Verify that
   \[
   u(x,t) = \frac{1}{2v} \int_{x-vt}^{x+vt} f(s) \, ds
   \]
   is a solution to the wave equation \( u_{tt} = v^2u_{xx} \), where \( v > 0 \) is constant and \( f \) is an arbitrary differentiable function. Also show that \( u_t(x,0) = f(x) \).

3. Suppose the solution to the 2 dimensional heat equation \( u_t = k\nabla^2 u \) only depends on the distance from the origin \( r = \sqrt{x^2 + y^2} \), that is, suppose \( u(x,y,t) \equiv v(r,t) \). Derive the PDE that \( v(r,t) \) satisfies.

4. Use the method of characteristics to solve
   \[
   \begin{cases}
   u_x + yu_y = 0, \\
u(0,y) = y^3
   \end{cases}
   \]

5. Solve the following initial value problem by the method of coordinates
   \[
   \begin{cases}
   u_t + u_x - 3u = t, \quad \text{for } x \in \mathbb{R}, \ t > 0 \\
u(x,0) = x^2
   \end{cases}
   \]
February 21st, 2020

Topics: Method of characteristics, classification of second order PDEs, method of coordinate transformation for hyperbolic PDEs.

1. Solve following problem by using the method of characteristics
   \[
   \begin{cases}
   u_t + u_x - 3u = t, & \text{for } x \in \mathbb{R}, \ t > 0 \\
   u(x, 0) = x^2
   \end{cases}
   \]

2. Use the method of characteristics to solve
   \[
   \begin{cases}
   (1 + t^2)u_t + u_x = 0, \\
   u(x, 0) = \sin(x)
   \end{cases}
   \]
   Sketch some of the characteristic curves for this PDE.

3. Solve following BVP
   \[
   \begin{cases}
   u_t + 2u_x = 0, & \text{for } x > 0, t > 0 \\
   u(x, 0) = e^{-x} \\
   u(0, t) = \frac{1}{1 + t^2}
   \end{cases}
   \]
   Sketch some of the characteristic curves and determine whether the solution \(u(x, t)\) continuous along the leading characteristic \(x = 2t\). What about its derivatives?

4. Method of coordinate transformation for 2nd order hyperbolic PDEs with constant coefficients.

5. Show that the PDE \(2u_{xx} + 5u_{xt} + 3u_{tt} = 0\) is hyperbolic. Then use the method of coordinate transformation to solve
   \[
   \begin{cases}
   2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \\
   u(x, 0) = 0 \\
   u_t(x, 0) = xe^{-x^2}
   \end{cases}
   \]

February 28th, 2020

Topics: Operator factorization, wave equation and diffusion equation on unbounded domains.

1. Find the general solution of
   \[
   3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)
   \]  \hspace{1cm}(6)

   \textit{Hint:} Factor the operator \(L = 3\partial_t^2 + 10\partial_x\partial_t + 3\partial_x^2\) and reduce (6) to a system of first order PDEs.

2. Verify that
   \[
   u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s, \tau) dsd\tau
   \]  \hspace{1cm}(7)
   solves the inhomogeneous wave equation on the real line:

   \[
   \begin{cases}
   u_{tt} - c^2u_{xx} = f(x, t), & \text{for } -\infty < x < +\infty \\
   u(x, 0) = u_t(x, 0) = 0,
   \end{cases}
   \]
3. Use Duhamel’s principle to write a formula for the solution to the inhomogeneous heat equation

\[
(i) : \begin{cases}
    u_t - ku_{xx} = f(x, t), & \text{for } -\infty < x < +\infty \\
    u(x, 0) = 0,
\end{cases}
\]

where \( f(x, t) = \sin(x) \) for \( |x| < l \) and \( f(x, t) = 0 \) for \( |x| > l \). (\( l > 0 \) is a constant).

\[\text{March 6th, 2020}\]

Topics: Midterm review.

1. Consider the following initial value problem

\[
\begin{cases}
    u_t + u_x - 3u = t, & \text{for } x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) = x^2
\end{cases}
\]

Introduce a new dependent variable \( v(x, t) = u(x, t)e^{-3t} \), write the PDE in the new variable and solve it, and then solve the original PDE using this transformation.

2. Show that the PDE \( 2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \) is hyperbolic. Then use the method of coordinate transformation to solve

\[
\begin{cases}
    2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \\
    u(x, 0) = 0 \\
    u_t(x, 0) = xe^{-x^2}
\end{cases}
\]

3. In class you used Duhamel’s principle to show that the general solution to the inhomogeneous wave equation

\[
(i) : \begin{cases}
    u_{tt} - c^2u_{xx} = f(x, t), & \text{for } -\infty < x < +\infty \\
    u(x, 0) = u_t(x, 0) = 0,
\end{cases}
\]

is given by

\[
u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s, \tau) dsd\tau
\]

Derive the same formula by applying the method of coordinate transformation to problem \( i \).

4. Solve the heat equation \( u_t - ku_{xx} = 0 \) on the real line with initial data \( u(x, 0) = e^{3x} \):

\[\text{March 27th, 2020}\]

Topics: Midterm Solutions.
### April 3rd, 2020

Topics: Wave equation and Heat equation on bounded domain, Separation of Variables, Dirichlet Boundary, Neumann Boundary.

1. Consider the IBVP for the 1-D wave equation on the interval $(0, L)$ with Dirichlet boundary data:

   \[
   \begin{aligned}
   \begin{cases}
   u_{tt} - c^2 u_{xx} = 0, & \text{for } 0 < x < +L \\
   u(0, t) = u(L, t) = 0, \\
   u(x, 0) = g(x), \\
   u_t(x, 0) = h(x)
   \end{cases}
   \end{aligned}
   \]

   Solve this Problem by Separation of Variables.

2. Consider the heat equation in 1-D on the interval $(0, \pi)$ with homogeneous Neumann boundary conditions:

   \[
   \begin{aligned}
   \begin{cases}
   u_t - 4u_{xx} = 0, & \text{for } 0 < x < +\pi \\
   u_x(0, t) = u_x(\pi, t) = 0, \\
   u(x, 0) = x
   \end{cases}
   \end{aligned}
   \]

   Solve this problem by Separation of Variables.

### April 10th, 2020

Topics: Separation of variables, Mixed Dirichlet-Neumann Boundary, Periodic Boundary, symmetries and heat equation on the half line.

1. Consider the heat equation in 1-D the interval $(0, L)$ with mixed Dirichlet-Neumann boundary conditions:

   \[
   \begin{aligned}
   \begin{cases}
   u_t - ku_{xx} = 0, & \text{for } 0 < x < L \\
   u(0, t) = u_x(L, t) = 0, \\
   u(x, 0) = f(x)
   \end{cases}
   \end{aligned}
   \]

   Solve this problem by Separation of Variables and express the solution by using Fourier Series.

2. Consider the heat equation on the interval $(0, L)$ with periodic boundary conditions:

   \[
   \begin{aligned}
   \begin{cases}
   u_t - ku_{xx} = 0, & \text{for } 0 < x < L \\
   u(0, t) = u(L, t), \quad u_x(0, t) = u_x(L, t), \\
   u(x, 0) = f(x)
   \end{cases}
   \end{aligned}
   \]

   Solve this by Separation of Variables and express the solution by using Fourier Series.
3. Consider the heat equation on the real line

\[ u_t - ku_{xx} = 0, \quad -\infty < x < +\infty, \quad t > 0, \quad u(x, 0) = \phi(x) \] \tag{9}

Show that if \( \phi(x) \) is an odd function, then \( u(-x, t) = -u(x, t) \) so that \( u(0, t) = 0 \). In the same way, show that if \( \phi(x) \) is an even function, then \( u(-x, t) = u(x, t) \) so that \( u_x(0, t) = 0 \). Use these facts to solve the heat equation on the half line with Dirichlet and Neumann Boundary Conditions, that is

\[
\begin{cases}
  u_t - ku_{xx} = 0, & \text{for } 0 < x < +\infty \\
  u(0, t) = 0, \\
  u(x, 0) = f(x),
\end{cases}
\]

and

\[
\begin{cases}
  u_t - ku_{xx} = 0, & \text{for } 0 < x < +\infty \\
  u_x(0, t) = 0, \\
  u(x, 0) = g(x).
\end{cases}
\]

4. Consider the eigenvalue problem \( X''(x) + \lambda X(x) = 0 \) for \( 0 < x < L \). For each of the following BC's, find the eigenvalues and eigenfunctions \( \{\lambda_n, X_n\} \) for the above BVP.

- Dirichlet: \( X(0) = X(L) = 0 \)
- Neumann: \( X'(0) = X'(L) = 0 \)
- Mixed: \( X(0) = X'(L) = 0 \)
- Periodic: \( X(0) = X(L), \ X'(0) = X'(L) \)

April 17th, 2020

Topics: Heat and Wave equation with inhomogeneous boundary conditions and sources.

1. Solve the heat equation with a source and nonzero IC

\[
\begin{cases}
  w_t - kw_{xx} = 2e^{-t}(1 - x), & 0 < x < 1, \ t > 0 \\
  w(0, t) = w(1, t) = 0, \ t > 0 \\
  w(x, 0) = x^2 + x, & 0 < x < 1.
\end{cases}
\]

2. Solve the heat equation with inhomogeneous BCs

\[
\begin{cases}
  u_t - ku_{xx} = 0, & 0 < x < 1, \ t > 0 \\
  u(0, t) = 2e^{-t}, \ u(1, t) = 1, \\
  u(x, 0) = x^2, & 0 < x < 1.
\end{cases}
\]

Note that for the diffusion equation we can always make a transformation of the dependent function to force zero boundary conditions at the expense of introducing a source term in the PDE. Use the superposition principle to reduce this problem to problem (1).
3. (Problem 4.7.9 in APDE) Solve the wave equation on \([0, 1]\) with inhomogeneous boundary data and source term

\[
\begin{aligned}
&u_{tt} - c^2 u_{xx} = q, \quad 0 < x < 1, \ t > 0 \\
u(0, t) = 0, \ u(1, t) = \sin(t), \ t > 0 \\
u(x, 0) = x(1 - x), \ u_t(x, 0) = 0, \ 0 < x < 1.
\end{aligned}
\]

April 24th, 2020

Topics: Laplace’s equation on a disk, heat equation on a 3D sphere and radially symmetric solution, advection-diffusion equation.

1. Laplace’s equation on a disk (Page 165 in PDE). Consider the problem

\[
\begin{aligned}
&u_{xx} + u_{yy} = 0, \quad \text{for} \ x^2 + y^2 < a, \\
u = h(\theta), \quad \text{for} \ x^2 + y^2 = a.
\end{aligned}
\]

Solve this by separation of variables. Hint: Separate the variables in polar coordinates as \(u(r, \theta) = R(r)\Theta(\theta)\). Remember that the Laplacian in polar coordinates is given by (can you show this?)

\[
u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.
\]

2. Solve the Poisson equation on a rectangle

\[
\begin{aligned}
&u_{xx} + u_{yy} = 1, \quad \text{for} \ 0 < x < a, \ 0 < y < b \\
u_x(0, y) = u_x(a, y) = 0 \\
u(x, 0) = u(x, b) = 0
\end{aligned}
\]

3. Find the solution for the advection-diffusion equation on the real line

\[
\begin{aligned}
&u_t + cu_x = ku_{xx}, \quad -\infty < x < +\infty, \ t > 0 \\
u(x, 0) = \phi(x).
\end{aligned}
\]

**Hint:** consider \(v(x, t) := u(x + ct, t)\) (See APDE Pages 40-43).