MATH-UA 263 Partial Differential Equations Recitation Summary

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Topics: verifying solution to a PDE, principle of continuum superposition and general solution to IVP for heat equation, general solution via integration, well-posedness.

1. Verify that the function

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \tag{1}$$

satisfies the heat equation $u_t = ku_{xx}$, for $t > 0$.

2. Verify that the general solution to the following IVP (Initial Value Problem)

$$\begin{cases} u_t = ku_{xx}, \quad \text{for } -\infty < x < +\infty,\ t > 0 \\ u(x, t = 0) = \phi(x) \end{cases}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy. \tag{2}$$

(For $t > 0$, use Problem (1); we also showed how this is an example of the principle of continuum superposition).

3. Solve the IVP in Problem (2) with initial condition $\phi(x) = e^{-x}$. In particular, use the general formula (2) to show that the solution is given by

$$u(x,t) = e^{kt-x}. \tag{3}$$

(In computing the integral, use the identity $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1$).

4. Find the general solution to the equation

$$u_{xt} + 3u_x = 1 \tag{4}$$

(we used the substitution $v = u_x$).

5. Consider the following Cauchy problems for Laplace's equation:

$$(i) \begin{cases} v_{xx} + v_{yy} = 0, \quad \text{for } -\infty < x < +\infty,\ y > 0 \\ v(x,0) = 0 \\ v_y(x,0) = 0 \end{cases}$$

$$(ii) \begin{cases} u_{xx} + u_{yy} = 0, \quad \text{for } -\infty < x < +\infty,\ y > 0 \\ u(x,0) = 0 \\ u_y(x,0) = e^{-\sqrt{n}} \sin(nx) \end{cases}$$

Show that (ii) is an ill-posed problem; in particular, show that it is not stable with respect to boundary data. (use that $v(x,y) = 0$ solves (i) and $u(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \sinh(ny)$ is a solution to (ii); look at what happens for large n).

February 14th, 2020

Topics: ODEs review, d'Alembert's formula, differential operators in polar coordiates, method of characteristics, method of coordinates, ill-posedness of backwards heat equation.

1. ODEs Review:

   - 1^{st} order equations, separable equations: $\frac{dy}{dx} = f(x)g(y)$. For example, solve IVP

   $$\begin{cases} y' = e^{-y}(2x-4), \\ y(5) = 0 \end{cases}$$

   - 1^{st} order linear equations, solvable by integrating factor method: $\frac{dy}{dx} + p(x)y = q(x)$. For example, solve IVP

   $$\begin{cases} y' = 5x - \frac{3y}{x}, \\ y(1) = 2 \end{cases}$$

   - 2^{nd} order equations with constant coefficients: $ay'' + by' + cy = d(x)$. For example, solve BVP

   $$\begin{cases} u'' - u = x, \quad \text{for} \ \ 0 < x < 2\pi \\ u(0) = u(2\pi) = 0. \end{cases}$$

   Also find the general solutions to two important examples: (i) $u'' + a^2u = 0$, and (ii) $u'' - a^2u = 0$ (a is constant).

2. (Homework 1, Problem 3) Verify that

   $$u(x,t) = \frac{1}{2v}\int_{x-vt}^{x+vt} f(s)ds \tag{5}$$

   is a solution to the wave equation $u_{tt} = v^2u_{xx}$, where $v > 0$ is constant and f is an arbitrary differentiable function. Also show that $u_t(x, 0) = f(x)$.

3. Suppose the solution to the 2 dimensional heat equation $u_t = k\nabla^2u$ only depends on the distance from the origin $r = \sqrt{x^2 + y^2}$, that is, suppose $u(x, y, t) \equiv v(r, t)$. Derive the PDE that $v(r, t)$ satisfies.

4. Use the method of characteristics to solve

   $$\begin{cases} u_x + yu_y = 0, \\ u(0, y) = y^3 \end{cases}$$

5. Solve the following initial value problem by the method of coordinates

   $$\begin{cases} u_t + u_x - 3u = t, \quad \text{for} \ \ x \in \mathbb{R}, \ t > 0 \\ u(x, 0) = x^2 \end{cases}$$

February 21st, 2020

Topics: Method of characteristics, classification of second order PDEs, method of coordinate transformation for hyperbolic PDEs.

1. Solve following problem by using the method of characteristics

$$\begin{cases} u_t + u_x - 3u = t, \quad \text{for } \; x \in \mathbb{R},\; t > 0 \\ u(x,0) = x^2 \end{cases}$$

2. Use the method of characteristics to solve

$$\begin{cases} (1+t^2)u_t + u_x = 0, \\ u(x,0) = \sin(x) \end{cases}$$

   Sketch some of the characteristic curves for this PDE.

3. Solve following BVP

$$\begin{cases} u_t + 2u_x = 0, \quad \text{for } \; x > 0, t > 0 \\ u(x,0) = e^{-x} \\ u(0,t) = \frac{1}{1+t^2} \end{cases}$$

   Sketch some of the characteristic curves and determine whether the solution $u(x,t)$ continuous along the leading characteristic $x = 2t$. What about its derivatives?

4. Method of coordinate transformation for 2^{nd} order hyperbolic PDEs with constant coefficients.

5. Show that the PDE $2u_{xx} + 5u_{xt} + 3u_{tt} = 0$ is hyperbolic. Then use the method of coordinate transformation to solve

$$\begin{cases} 2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = xe^{-x^2} \end{cases}$$

February 28th, 2020

Topics: Operator factorization, wave equation and diffusion equation on unbounded domains.

1. Find the general solution of

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t) \tag{6}$$

   Hint: Factor the operator $\mathcal{L} = 3\partial_t^2 + 10\partial_x\partial_t + 3\partial_x^2$ and reduce (6) to a system of first order PDEs.

2. Verify that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau \tag{7}$$

   solves the inhomogeneous wave equation on the real line:

$$(i): \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \quad \text{for } \; -\infty < x < +\infty \\ u(x,0) = u_t(x,0) = 0, \end{cases}$$

3. Use Duhamel's principle to write a formula for the solution to the inhomogeneous heat equation

$$(i):\begin{cases} u_t - ku_{xx} = f(x,t), \text{ for } -\infty < x < +\infty \\ u(x,0) = 0, \end{cases}$$

where $f(x,t) = \sin(x)$ for $|x| < l$ and $f(x,t) = 0$ for $|x| > l$. ($l > 0$ is a constant).

March 6th, 2020

Topics: Midterm review.

1. Consider the following initial value problem

$$\begin{cases} u_t + u_x - 3u = t, \quad \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = x^2 \end{cases}$$

Introduce a new dependent variable $v(x,t) = u(x,t)e^{-3t}$, write the PDE in the new variable and solve it, and then solve the original PDE using this transformation.

2. Show that the PDE $2u_{xx} + 5u_{xt} + 3u_{tt} = 0$ is hyperbolic. Then use the method of coordinate transformation to solve

$$\begin{cases} 2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = xe^{-x^2} \end{cases}$$

3. In class you used Duhamel's principle to show that the general solution to the inhomogeneous wave equation

$$(i):\begin{cases} u_{tt} - c^2u_{xx} = f(x,t), \text{ for } -\infty < x < +\infty \\ u(x,0) = u_t(x,0) = 0, \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau \tag{8}$$

Derive the same formula by applying the method of coordinate transformation to problem (i).

4. Solve the heat equation $u_t - ku_{xx} = 0$ on the real line with initial data $u(x,0) = e^{3x}$:

March 27th, 2020

Topics: Midterm Solutions.

April 3rd, 2020

Topics: Wave equation and Heat equation on bounded domain, Separation of Variables, Dirichlet Boundary, Neumann Boundary.

1. Consider the IBVP for the 1-D wave equation on the interval $(0, L)$ with Dirichlet boundary data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \text{ for } 0 < x < +L \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = g(x), \\ u_t(x,0) = h(x) \end{cases}$$

   Solve this Problem by Separation of Variables.

2. Consider the heat equation in 1-D on the interval $(0, \pi)$ with homogeneous Neumann boundary conditions

$$\begin{cases} u_t - 4u_{xx} = 0, \text{ for } 0 < x < +\pi \\ u_x(0,t) = u_x(\pi,t) = 0, \\ u(x,0) = x, \end{cases}$$

   Solve this problem by Separation of Variables.

April 10th, 2020

Topics: Separation of variables, Mixed Dirichlet-Neumann Boundary, Periodic Boundary, symmetries and heat equation on the half line.

1. Consider the heat equation in 1-D the interval $(0, L)$ with mixed Dirichlet-Neumann boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0, \text{ for } 0 < x < L \\ u(0,t) = u_x(L,t) = 0, \\ u(x,0) = f(x), \end{cases}$$

   Solve this problem by Separation of Variables and express the solution by using Fourier Series.

2. Consider the heat equation on the interval $(0, L)$ with periodic boundary conditions:

$$\begin{cases} u_t - ku_{xx} = 0, \text{ for } 0 < x < L \\ u(0,t) = u(L,t), \quad u_x(0,t) = u_x(L,t), \\ u(x,0) = f(x), \end{cases}$$

   Solve this by Separation of Variables and express the solution by using Fourier Series.

3. Consider the heat equation on the real line

$$u_t - ku_{xx} = 0, \ -\infty < x < +\infty, \ t > 0, \quad u(x,0) = \phi(x) \tag{9}$$

Show that if $\phi(x)$ is an odd function, then $u(-x,t) = -u(x,t)$ so that $u(0,t) = 0$. In the same way, show that if $\phi(x)$ is an even function, then $u(-x,t) = u(x,t)$ so that $u_x(0,t) = 0$. Use these facts to solve the heat equation on the half line with Dirichlet and Neumann Boundary Conditions, that is

$$\begin{cases} u_t - ku_{xx} = 0, \ \text{for} \ \ 0 < x < +\infty \\ u(0,t) = 0, \\ u(x,0) = f(x), \end{cases}$$

and

$$\begin{cases} u_t - ku_{xx} = 0, \ \text{for} \ \ 0 < x < +\infty \\ u_x(0,t) = 0, \\ u(x,0) = g(x). \end{cases}$$

4. Consider the eigenvalue problem $X''(x) + \lambda X(x) = 0$ for $0 < x < L$. For each of the following BC's, find the eigenvalues and eigenfunctions $\{\lambda_n, X_n\}$ for the above BVP.

- Dirichlet: $X(0) = X(L) = 0$
- Neumann: $X'(0) = X'(L) = 0$
- Mixed: $X(0) = X'(L) = 0$
- Periodic: $X(0) = X(L), \ X'(0) = X'(L)$

April 17th, 2020

Topics: Heat and Wave equation with inhomogeneous boundary conditions and sources.

1. Solve the heat equation with a source and nonzero IC

$$\begin{cases} w_t - kw_{xx} = 2e^{-t}(1-x), \quad 0 < x < 1, \ t > 0 \\ w(0,t) = w(1,t) = 0, \ t > 0 \\ w(x,0) = x^2 + x,, \quad 0 < x < 1. \end{cases}$$

2. Solve the heat equation with inhomogeneous BCs

$$\begin{cases} u_t - ku_{xx} = 0, \quad 0 < x < 1, \ t > 0 \\ u(0,t) = 2e^{-t}, \quad u(1,t) = 1, \\ u(x,0) = x^2, \quad 0 < x < 1. \end{cases}$$

Note that for the diffusion equation we can always make a transformation of the dependent function to force zero boundary conditions at the expense of introducing a source term in the PDE. Use the superposition principle to reduce this problem to problem (1).

3. (Problem 4.7.9 in APDE) Solve the wave equation on $[0,1]$ with inhomogeneous boundary data and source term

$$\begin{cases} u_{tt} - c^2 u_{xx} = q, \quad 0 < x < 1,\ t > 0 \\ u(0,t) = 0, \quad u(1,t) = \sin(t), \quad t > 0 \\ u(x,0) = x(1-x),\ u_t(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

April 24th, 2020

Topics: Laplace's equation on a disk, Poisson equation, advection-diffusion equation.

1. Laplace's equation on a disk (Page 165 in PDE). Consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad \text{for } \ x^2 + y^2 < a, \\ u = h(\theta), \quad \text{for } \ x^2 + y^2 = a. \end{cases}$$

   Solve this by separation of variables. *Hint*: Separate the variables in poolar coordinates as $u(r,\theta) = R(r)\Theta(\theta)$. Remember that the Laplacian in poolar coordinates is given by (can you show this?)

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{10}$$

2. Solve the Poisson equation on a rectangle

$$\begin{cases} u_{xx} + u_{yy} = 1, \quad \text{for } \ 0 < x < a,\ 0 < y < b \\ u_x(0,y) = u_x(a,y) = 0 \\ u(x,0) = u(x,b) = 0 \end{cases}$$

May 1st, 2020

Topics: Review 1.

1. Find the solution for the advection-diffusion equation on the real line

$$\begin{cases} u_t + cu_x = ku_{xx}, & -\infty < x < +\infty,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

   Hint: consider $v(x,t) := u(x+ct,t)$ (See APDE Pages 40-43).

2. Solve the heat equation on $(0,1)$ with source term and inhomogeneous BCs

$$\begin{cases} u_t - ku_{xx} = f(x,t), & 0 < x < 1,\ t > 0 \\ u(0,t) = 2e^{-t}, \quad u(1,t) = 1, & t > 0 \\ u(x,0) = x^2, & 0 < x < 1. \end{cases}$$

   Hint: First, use the superposition principle to split this problem into simpler subproblems; then, use separation of variables.

3. Using separation of variables, find the solution to the elliptic PDE $-(u_{xx}+u_{yy})+2u = 0$ in the square domain $0 < x, y < 1$ with BCs $u(x,0) = u(x,1) = 0$ and $u(0,y) = (1-y)y$ and $u_x(1,y) = 0$.

May 8th, 2020

Topics: Review 2.

1. Solve the 2D heat equation with source and inhomogeneous BCs on a square domain

$$(P) : \begin{cases} u_t = u_{xx} + u_{yy} + f(x,y,t), & 0 < x < 1,\ 0 < y < 1,\ t > 0 \\ u(x,0,t) = u(x,1,t) = 0, \\ u(0,y,t) = ye^{-t}, \quad u_x(1,y,t) = 0, \\ u(x,y,0) = \psi(x,y). \end{cases}$$

   Hint: Split this problem into subproblems that can be handled by separation of variables / eigenfunction expansion methods. In particular, (Step 1) find eigenfunctions and eigenvalues of the Laplacian with homogeneous BCs, (Step 2) use superposition to split (P) into the following 2 subproblems:

   - (P1): Steady state version of (P) with <u>in</u>homogeneous BCs and no source term $\rightarrow$ Solve by separation of variables
   - (P2): Heat equation with (extra) source term and homogeneous BCs $\rightarrow$ Use Step 1 to solve by eigenfunction expansion.

2. Use separation of variables to solve the PDE

$$u_{tt} + 2u_t = u_{xx} \tag{11}$$

   on the strip $0 < x < 1$, $t > 0$ subject to BCs $u_x(0,t) = u_x(1,t) = 0$ and ICs $u(x,0) = 0$ and $u_t(x,0) = 1 - \cos(2\pi x)$.

3. Use the method of characteristics to solve the first order PDE $xu_t + tu_x = -xu$ where $u(x,0) = 1 - x$ and $x \geq 0$, $t \geq 0$. Sketch some characteristic curves. Can the solution be determined everywhere in the first quadrant? If no, for which values of x and t is the solution valid?