February 7th, 2020

Topics: verifying solution to a PDE, principle of continuum superposition and general solution to IVP for heat equation, general solution via integration, well-posedness.

1. Verify that the function
\[ G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \]
satisfies the heat equation \( u_t = ku_{xx} \), for \( t > 0 \).

2. Verify that the general solution to the following IVP (Initial Value Problem)
\[
\begin{aligned}
&\quad \begin{cases}
    u_t = ku_{xx}, & \text{for } -\infty < x < +\infty, \ t > 0 \\
    u(x, t = 0) = \phi(x)
\end{cases} \\
&\quad \begin{cases}
    u(x, t = 0) = \phi(x)
\end{cases}
\end{aligned}
\]
is given by
\[ u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy. \]  
(For \( t > 0 \), use Problem (1); we also showed how this is an example of the principle of continuum superposition).

3. Solve the IVP in Problem (2) with initial condition \( \phi(x) = e^{-x} \). In particular, use the general formula (2) to show that the solution is given by
\[ u(x,t) = e^{kt-x}. \]  
(In computing the integral, use the identity \( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1 \)).

4. Find the general solution to the equation
\[ u_{xt} + 3u_x = 1 \]  
(we used the substitution \( v = u_x \)).

5. Consider the following Cauchy problems for Laplace’s equation:
\[
\begin{cases}
    v_{xx} + v_{yy} = 0, & \text{for } -\infty < x < +\infty, \ y > 0 \\
    v(x, 0) = 0 \\
    v_y(x, 0) = 0 \\
    u_{xx} + u_{yy} = 0, & \text{for } -\infty < x < +\infty, \ y > 0 \\
    u(x, 0) = 0 \\
    u_y(x, 0) = e^{-\sqrt{\pi}} \sin(nx)
\end{cases}
\]
(i) \( v(x, 0) = 0 \)
(ii) \( u(x, 0) = 0 \)

Show that (ii) is an ill-posed problem; in particular, show that it is not stable with respect to boundary data. (use that \( v(x, y) = 0 \) solves (i) and \( u(x, y) = \frac{1}{n} e^{-\sqrt{\pi}} \sin(nx) \sinh(ny) \) is a solution to (ii); look at what happens for large \( n \)).
February 14th, 2020

Topics: ODEs review, d’Alembert’s formula, differential operators in polar coordinates, method of characteristics, method of coordinates, ill-posedness of backwards heat equation.

1. ODEs Review:
   - 1st order equations, separable equations: \( \frac{dy}{dx} = f(x)g(y) \). For example, solve IVP
     \[
     \begin{cases}
     y' = e^{-y}(2x - 4), \\
     y(5) = 0
     \end{cases}
     \]
   - 1st order linear equations, solvable by integrating factor method: \( \frac{du}{dx} + p(x)y = q(x) \). For example, solve IVP
     \[
     \begin{cases}
     y' = 5x - \frac{3y}{x}, \\
     y(1) = 2
     \end{cases}
     \]
   - 2nd order equations with constant coefficients: \( ay'' + by' + cy = d(x) \). For example, solve BVP
     \[
     \begin{cases}
     u'' - u = x, \text{ for } 0 < x < 2\pi \\
     u(0) = u(2\pi) = 0.
     \end{cases}
     \]
     Also find the general solutions to two important examples: (i) \( u'' + a^2u = 0 \), and (ii) \( u'' - a^2u = 0 \) \((a \text{ is constant})\).

2. (Homework 1, Problem 3) Verify that
   \[
   u(x,t) = \frac{1}{2v} \int_{x-\sqrt{vt}}^{x+\sqrt{vt}} f(s)ds
   \]
   is a solution to the wave equation \( u_{tt} = v^2 u_{xx} \), where \( v > 0 \) is constant and \( f \) is an arbitrary differentiable function. Also show that \( u_t(x,0) = f(x) \).

3. Suppose the solution to the 2 dimensional heat equation \( u_t = k\nabla^2 u \) only depends on the distance from the origin \( r = \sqrt{x^2 + y^2} \), that is, suppose \( u(x,y,t) \equiv v(r,t) \). Derive the PDE that \( v(r,t) \) satisfies.

4. Use the method of characteristics to solve
   \[
   \begin{cases}
   u_x + yu_y = 0, \\
   u(0, y) = y^3
   \end{cases}
   \]

5. Solve the following initial value problem by the method of coordinates
   \[
   \begin{cases}
   u_t + u_x - 3u = t, \text{ for } x \in \mathbb{R}, t > 0 \\
   u(x,0) = x^2
   \end{cases}
   \]
February 21th, 2020

Topics: Method of characteristics, classification of second order PDEs, method of coordinate transformation for hyperbolic PDEs.

1. Solve following problem by using the method of characteristics

\[
\begin{cases}
    u_t + u_x - 3u = t, & \text{for } x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) = x^2
\end{cases}
\]

2. Use the method of characteristics to solve

\[
\begin{cases}
    (1 + t^2)u_t + u_x = 0, \\
    u(x, 0) = \sin(x)
\end{cases}
\]
Sketch some of the characteristic curves for this PDE.

3. Solve following BVP

\[
\begin{cases}
    u_t + 2u_x = 0, & \text{for } x > 0, t > 0 \\
    u(x, 0) = e^{-x} \\
    u(0, t) = \frac{1}{1+t^2}
\end{cases}
\]
Sketch some of the characteristic curves and determine whether the solution \(u(x, t)\) continuous along the leading characteristic \(x = 2t\). What about its derivatives?

4. Method of coordinate transformation for 2\textsuperscript{nd} order hyperbolic PDEs with constant coefficients.

5. Show that the PDE \(2u_{xx} + 5u_{xt} + 3u_{tt} = 0\) is hyperbolic. Then use the method of coordinate transformation to solve

\[
\begin{cases}
    2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \\
    u(x, 0) = 0 \\
    u_t(x, 0) = xe^{-x^2}
\end{cases}
\]