# PDE Spring 2016

## PDE Classification

$u \equiv u(x,y)$

Consider the general second-order linear PDE

$$a u_{xx} + 2b u_{xy} + c u_{yy} + p u_x + q u_y + r u = f$$

where

$$a \equiv a(x,y) \text{ etc.}$$

For PDEs the nature of the equation is (typically) dominated by the highest derivative, so let's delete the first-order terms

$$\mathcal{L}u = f(x,y)$$

$$\mathcal{L} = a\,\partial_{xx} + 2b\,\partial_x\partial_y + c\,\partial_{yy}$$

The nature of the PDE is determined by the

discriminant $d = b^2 - ac$

1 $\boxed{b^2 - ac > 0}$ — equation is hyperbolic

This means that with a simple change of coordinates it can be transformed into the equation

$$\mathcal{L}' u(s,t) = g$$

where

$$\mathcal{L}' = \partial_{st}$$

has only a mixed second derivative

2 $\boxed{b^2 - ac = 0 \,, \; a > 0}$

Eq. is parabolic

meaning that it can be transformed into the equation

$$\mathcal{L}' u(s,t) = g$$

$$\mathcal{L}' = \partial_{ss}$$

so there is only a single second order derivative with no mixed derivatives

3 $\boxed{b^2 - ac < 0}$ - elliptic

$$\mathcal{L}' = \partial_{ss} + \partial_{tt}$$

so there is now a Laplacian appearing

The canonical form is a form into which every second-order linear equation can be transformed into by a change of independent variables (i.e, of coordinates):

(1) Canonical hyperbolic equation

$$u_{st} + G(s, t, u, u_s, u_t) = 0$$

Example: the wave equation $u_{tt} = \nabla^2 u$

(2) Canonical parabolic equation

$$u_{ss} + H(s, t, u, u_s, u_t) = 0$$

Ex: the heat equation $u_t = \nabla^2 u$

(3) Canonical elliptic eq.

$$u_{ss} + u_{tt} + F(s, t, u, u_s, u_t) = 0$$

Ex: Laplace equation $\nabla^2 u = 0$

We need to study and understand each type of equation separately, but it is OK to choose one sample prototype equation from each category:

$$\begin{cases} \text{Wave eq} & u_{tt} = \nabla^2 u \text{ - hyperbolic} \\ \text{heat eq.} & u_t = \nabla^2 u \text{ - parabolic} \\ \text{Laplace eq.} & \nabla^2 u = 0 \text{ - elliptic} \end{cases}$$

One way to think of this is as trying to factorize the operator

$$\mathcal{L} = a\partial_x^2 + 2b\,\partial_{xy} + c\,\partial_y^2$$

just like one would factorize a quadratic polynomial: there are either two distinct real roots, one degenerate real root, or two complex-conjugate roots

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We would write

$$a^2 - b^2 = (a-b)(a+b)$$

so

$$\mathcal{L}u = u_{tt} - u_{xx} \quad \text{(wave eq.)}$$

$$\mathcal{L} = \partial_{tt} - \partial_{xx} = \partial_t^2 - \partial_x^2 =$$

$$(\partial_t - \partial_x)(\partial_t + \partial_x) = \mathcal{L}^- \mathcal{L}^+$$

$$\begin{cases} \mathcal{L}^- = \partial_t - \partial_x \\ \mathcal{L}^+ = \partial_t + \partial_x \end{cases}$$

What this means is that

$$\mathcal{L}u = \mathcal{L}^-(\mathcal{L}^+ u) = \mathcal{L}^+(\mathcal{L}^- u)$$

Exercise: show this explicitly

Observe $\mathcal{L}^-$ and $\mathcal{L}^+$ are associated with the advection equations with speed $\pm 1$.
The advection equation is a degenerate kind of hyperbolic equation

<u>Hyperbolic</u>: (wave-like)

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)(\gamma \partial_x + \delta \partial_y)$$

with $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$

and $\alpha, \beta, \gamma, \delta$ are real

<u>Parabolic</u>: (diffusive)

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)^2$$

<u>Elliptic</u> (static)

$$\mathcal{L} = (\alpha \partial_x + i\beta \partial_y)(\alpha \partial_x - i\beta \partial_y)$$

<u>not</u> most general

$$= \alpha^2 \partial_x^2 + \beta^2 \partial_y^2$$

## Higher-order PDEs

$$u_{ttt} + u_{txx} + u\,u_{ttx} + u\,u_{xxx} = 0 =$$

$$(\partial_t + u\partial_x)(u_{tt} + u_{xx})$$

$$= \underbrace{(\partial_t + u\partial_x)}_{\text{hyperbolic piece}}\underbrace{(\partial_t + i\partial_x)(\partial_t - i\partial_x)}_{\text{elliptic piece}}u$$

so this is a mixed hyperbolic-elliptic equation

But this classification is not as useful anymore...

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## Examples

discriminant $= -12$ elliptic

(a) $u_{xx} - 6u_{xy} + 12u_{yy} = 0$

Introduce change of variables

$$\begin{cases} s = \alpha x + \beta y \\ t = \gamma x + \delta y \end{cases} \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \text{ unknown}$$

$$u_x = \alpha u_s + \gamma u_t$$

$$\Rightarrow u_{xx} = \alpha(\alpha u_{ss} + \gamma u_{st}) + \gamma(\alpha u_{st} + \gamma u_{tt})$$

$$= \alpha^2 u_{ss} + 2\alpha\gamma u_{st} + \gamma^2 u_{tt}$$

One can grind through algebra for $u_{tt}$ and $u_{st}$ (<u>tedious</u>)

A transformation that works

$$\begin{cases} s = x + \frac{1}{4} y \\ t = \frac{\sqrt{3}}{12} y \end{cases} \Rightarrow u_{ss} + u_{tt} = 0$$

and $u_{st}$ term disappears!

(10)

(b) $$2u_{xx} - 4u_{xy} + u_x = 0$$

(hyperbolic)

$$\begin{cases} s = 2x + y \\ t = y \end{cases}$$

Now we get

$$u_{st} = \frac{1}{4} u_s$$

so only the mixed term remains

(c)

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

(parabolic)

$$\begin{cases} s = x \\ t = x - y \end{cases} \Rightarrow$$

$$u_{ss} = 0$$