PDE Spring 2016

PDE Classification

\[ u = u(x, y) \]

Consider the general second-order linear PDE

\[ a u_{xx} + 2b u_{xy} + c u_{yy} \]
\[ + p u_x + q u_y + r u = f \]

where

\[ a = a(x, y) \quad \text{etc.} \]

For PDEs the nature of the equation is (typically) dominated by the highest derivative, so let's delete the first-order terms.
\[ Lu = f(x, y) \]
\[ L = a \partial_{xx} + 2b \partial_{xy} + c \partial_{yy} \]

The nature of the PDE is determined by the discriminant \( d = b^2 - ac \)

\( \square \) \[ b^2 - ac > 0 \] - equation is hyperbolic

This means that with a simple change of coordinates it can be transformed into the equation

\[ L' u(s,t) = 0 \]

Where

\[ L' = \partial_{st} \]

has only a mixed second derivative
2. \( b^2 - ac = 0, \ a > 0 \)

Eq. is \underline{parabolic}

meaning that it can be transformed into the equation

\[ x' n(s,t) = g \]

\[ x' = \partial ss \]

so there is only a single second order derivative with no mixed derivatives

3. \( b^2 - ac < 0 \) \underline{elliptic}

\[ x' = \partial ss + \partial tt \]

so there is now a Laplacian appearing
The canonical form is a form into which every second-order linear equation can be transformed into by a change of independent variables (i.e., of coordinates).

1. **Canonical hyperbolic equation**
   \[ U_{st} + G(s, t, u, u_s, u_t) = 0 \]

   Example: the wave equation
   \[ U_{tt} = \Delta^2 U \]

2. **Canonical parabolic equation**
   \[ U_{ss} + H(s, t, u, u_s, u_t) = 0 \]

   Ex: the heat equation
   \[ U_t = \Delta^2 U \]

3. **Canonical elliptic eq.**
   \[ U_{ss} + U_{tt} + F(s, t, u, u_s, u_t) = 0 \]

   Ex: Laplace equation
   \[ \Delta^2 u = 0 \]
We need to study and understand each type of equation separately, but it is OK to choose one sample prototype equation from each category:

\[
\begin{align*}
\text{Wave eq. } & \quad U_{tt} = \nabla^2 U - \text{hyperbolic} \\
\text{Heat eq. } & \quad U_t = \nabla^2 U - \text{parabolic} \\
\text{Laplace eq. } & \quad \nabla^2 U = 0 - \text{elliptic}
\end{align*}
\]

One way to think of this is as trying to factorize the operator

\[ L = a I_x^2 + 2b I_{xy} + c I_y^2 \]

just like one would factorize a quadratic polynomial; there are either two distinct real roots, one degenerate real root, or two complex-conjugate roots.
We would write
\[ a^2 - b^2 = (a-b)(a+b) \]
so
\[ \mathcal{L}u = u_{tt} - u_{xx} \quad \text{(wave eq.)} \]
\[ \mathcal{L} = \partial_{tt} - \partial_{xx} = \partial_{t}^2 - \partial_{x}^2 = (\partial_{t} - \partial_{x})(\partial_{t} + \partial_{x}) = \mathcal{L}^- \mathcal{L}^+ \]
\[
\begin{cases} 
\mathcal{L}^- = \partial_{t} - \partial_{x} \\
\mathcal{L}^+ = \partial_{t} + \partial_{x} 
\end{cases}
\]
What this means is that
\[ \mathcal{L}u = \mathcal{L}^- (\mathcal{L}^+ u) = \mathcal{L}^+ (\mathcal{L}^- u) \]
Exercise: Show this explicitly

Observe \( \mathcal{L}^- \) and \( \mathcal{L}^+ \) are associated with the advection equations with speed \( \pm 1 \).

The advection equation is a degenerate kind of hyperbolic equation.
Hyperbolic: (wave-like)

\[ \lambda = (\alpha \partial_x + \beta \partial_y)(\beta \partial_x + \delta \partial_y) \]

with \( \beta \neq \delta \)

and \( \lambda, \beta, \delta, \delta \) are real

Parabolic: (diffusive)

\[ \lambda = (\alpha \partial_x + \beta \partial_y)^2 \]

Elliptic: (static)

\[ \lambda = (\alpha \partial_x + i\beta \partial_y)(\alpha \partial_x - i\beta \partial_y) \]

\[ \lambda \neq \text{most general} \]

\[ \lambda = \alpha^2 \partial_x^2 + \beta^2 \partial_y^2 \]
Higher-order PDEs

\[ \begin{align*}
& u_{ttt} + u_{txx} + uu_{ttt} + \\
& uu_{xxx} = 0 = \\
& (\Theta_t + u\Theta_x)(u_{tt} + u_{xx})
\end{align*} \]

= \left(\Theta_t + u\Theta_x\right)\left(\Theta_t + i\Theta_x\right)\left(\Theta_t - i\Theta_x\right)u

\text{hyperbolic piece} \quad \text{elliptic piece}

So this is a mixed hyperbolic-elliptic equation

But this classification is not as useful anymore...
Examples (discriminant = -12) elliptic

\[ u_{xx} - 6u_{xy} + 12u_{yy} = 0 \]

Introduce change of variables
\[
\begin{align*}
S &= \alpha x + \beta y \\
T &= \gamma x + \delta y \\
\end{align*}
\]

\[ u_x = \alpha u_S + \gamma u_T \]

\[ u_{xx} = \alpha (\alpha u_{SS} + \gamma u_{ST}) \]

\[ + \gamma (\alpha u_{ST} + \delta u_{TT}) \]

\[ = \alpha^2 u_{SS} + 2\alpha\gamma u_{ST} + \gamma^2 u_{TT} \]

One can grind through algebra for \( u_{TT} \) and \( u_{ST} \) (tedious)

A transformation that works

\[
\begin{align*}
S &= x + \frac{1}{4} y \\
T &= \frac{\sqrt{3}}{12} y \\
\end{align*}
\]

\[ U_{SS} + U_{TT} = 0 \]

and \( U_{ST} \) term disappears!
\( 2u_{xx} - 4u_{xy} + u_x = 0 

(\text{hyperbolic}) \\

\begin{align*}
S &= 2x + y \\
t &= y
\end{align*}

Now we get

\[ u_{st} = \frac{1}{4} u_s \]

so only the mixed term remains

\( u_{xx} + 2u_{xy} + u_{yy} = 0 

(\text{parabolic}) \\

\begin{align*}
S &= x \\
t &= x - y
\end{align*}

\[ u_{ss} = 0 \]