(1) PDE Spring 2018, A. Donev

Nonlinear Advection

When we discussed traffic flow, we came up with a nonlinear advection equation of the form

$$u_t + a(u)\, u_x = 0$$

$\uparrow$ nonlinear function

Let's look for the characteristics and use time to parametrize them:

$x(t) \leftarrow$ one characteristic curve

$w = u(x(t), t)$ and we ~~w and~~ $\frac{dw}{dt} = 0$

$$\frac{du}{dt}(x(t), t) = u_x \frac{dx}{dt} + u_t = 0$$

Recall $a(u)\, u_x + u_t = 0$

$\Rightarrow dx/dt = a(u)$

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But, we don't know u
so we don't know $a(u)$

But, along a characteristic
$\frac{du}{dt} = 0$ so $u(x(t), t) = \underline{\underline{const}}$
Therefore $a(u) = const. = a(u_0)$

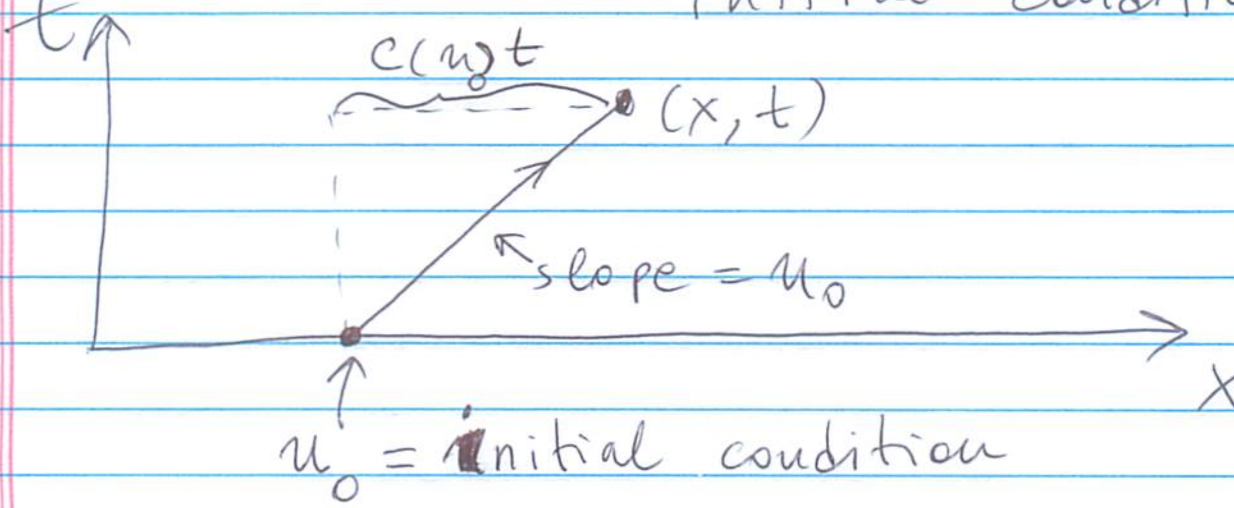
$$\Rightarrow \quad \frac{dx}{dt} = a(u_0) = const$$

$$x = a(u_0)\, t + \text{int. const.}$$

a) <u>Characteristics are straight lines</u>
b) u is constant along characteristics
c) The slope is determined by the initial value of u on the boundary (i.e. the initial conditions)

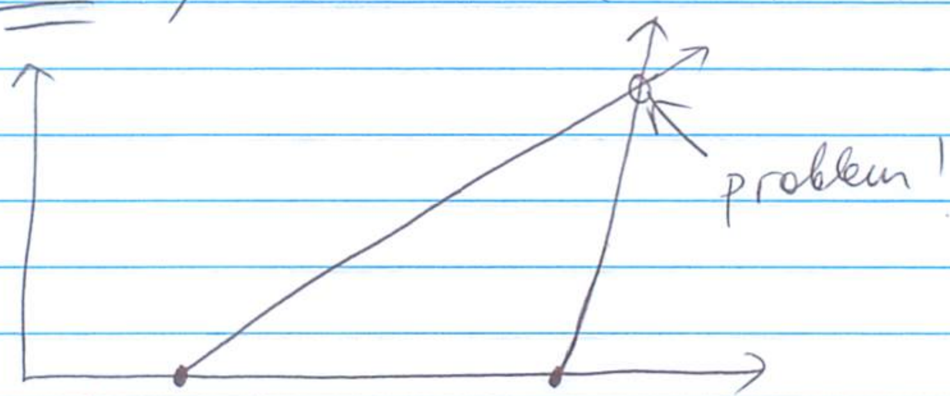
This is enough to "solve" the PDE graphically

$$\boxed{u(x,t) = \underbrace{u(x - c(u)t, 0)}_{\text{initial condition}}}$$



Note: We already showed earlier that this implicit solution satisfies the PDE

<u>But</u>, there is a twist!



$$c(u_0(x_1)) < c(u_0(x_2))$$

<u>Characteristics can intersect</u>

for nonlinear equations

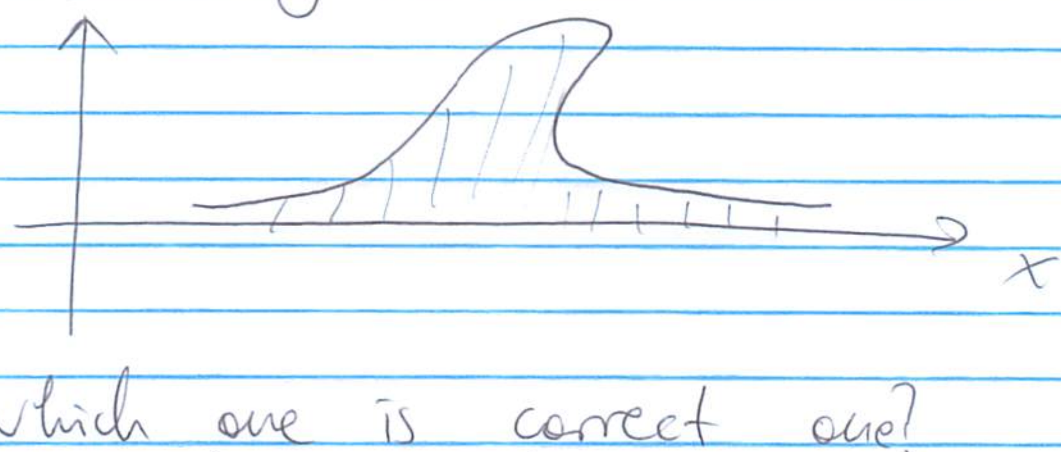
We get a contradiction:
two different values for the same point, so $u(x,t)$ cannot remain a function past the first point in time when two characteristics cross

What to do? Options include:

a) Declare equation ill-posed for all time
b) Extend the notion of a solution to include discontinuous solutions:

Theory of ~~[illegible]~~ shocks

c) Allow multi-valued (not a function solution), such as breaking waves on beach:



Which one is correct one?
Depends on physics (not math)

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Let's take traffic flow as an example based on notes of Prof. Esteban Tabak

$$\begin{cases} u_t + \varphi_x = 0 \\ \varphi = u(1-u) \end{cases}$$

$$u_t + \varphi' u_x = 0$$

$$\text{so} \quad a(u) \equiv \varphi'$$

<u>Characteristic</u> curve $X(t)$ satisfies

$$\frac{dx}{dt} = \varphi'$$

For advection

$$\varphi = u \cdot c(u)$$

where c is average car speed and depends on u

$$\cancel{\frac{dx}{dt} = \frac{d}{du}(c\,u) = u + c\,u}$$

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$$\frac{dx}{dt} = \varphi' = \frac{d}{du}\left(u\,c(u)\right)$$

$$= c(u) + u\,\frac{dc}{du}(u)$$

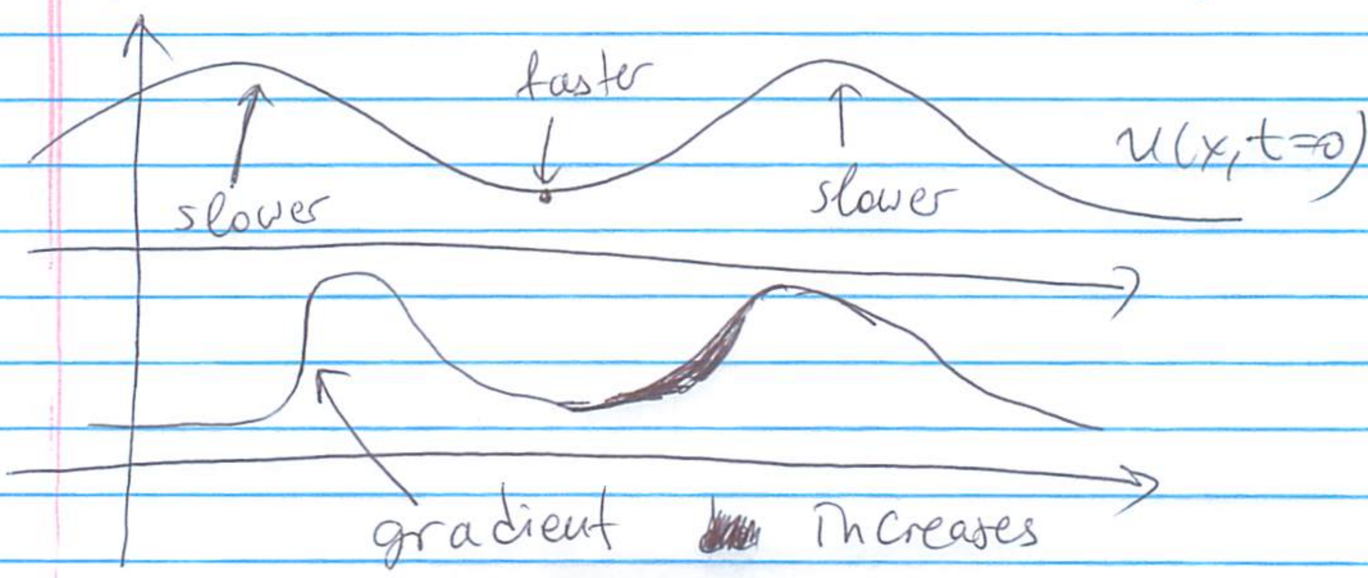
But c decreases with u.
(cars go slower at higher densities)

So

$$\boxed{\frac{dx}{dt} < c(u)}$$ for traffic

In other words, the traffic moves faster than characteristics, i.e., faster than information $\Rightarrow$

information on the state of traffic comes to the driver from in-front $\Rightarrow$ enables driving!



$\varphi = u(1-u)$ $\qquad$ $u=1$ is max (jamming density)

$\varphi' = 1-2u$

$\Rightarrow \varphi' = 0$ if $u = \frac{1}{2}$

we see a discontinuity forming with infinite slope – a <u>shock</u> (traffic jam)

Observe that this solution developed <u>spontaneously</u> out of a pretty nice initial condition

(that's why traffic jams make no sense sometimes)

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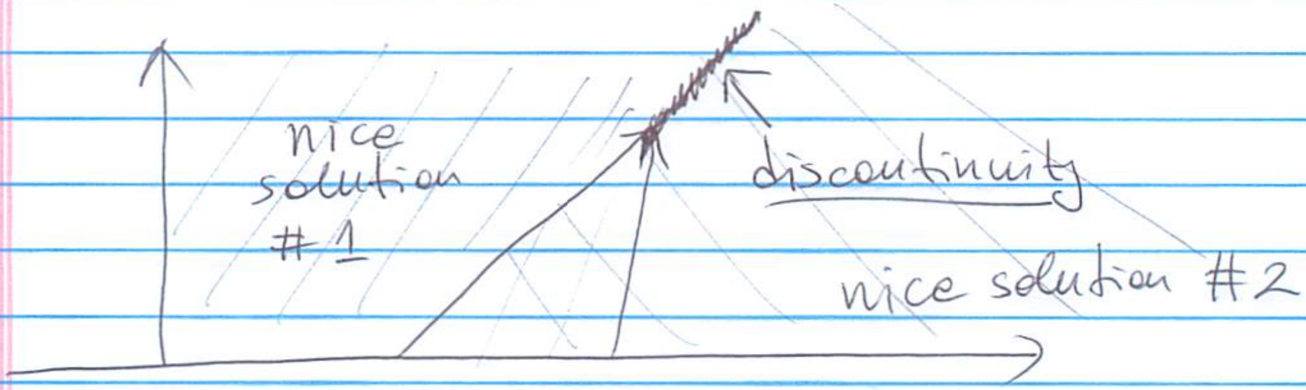
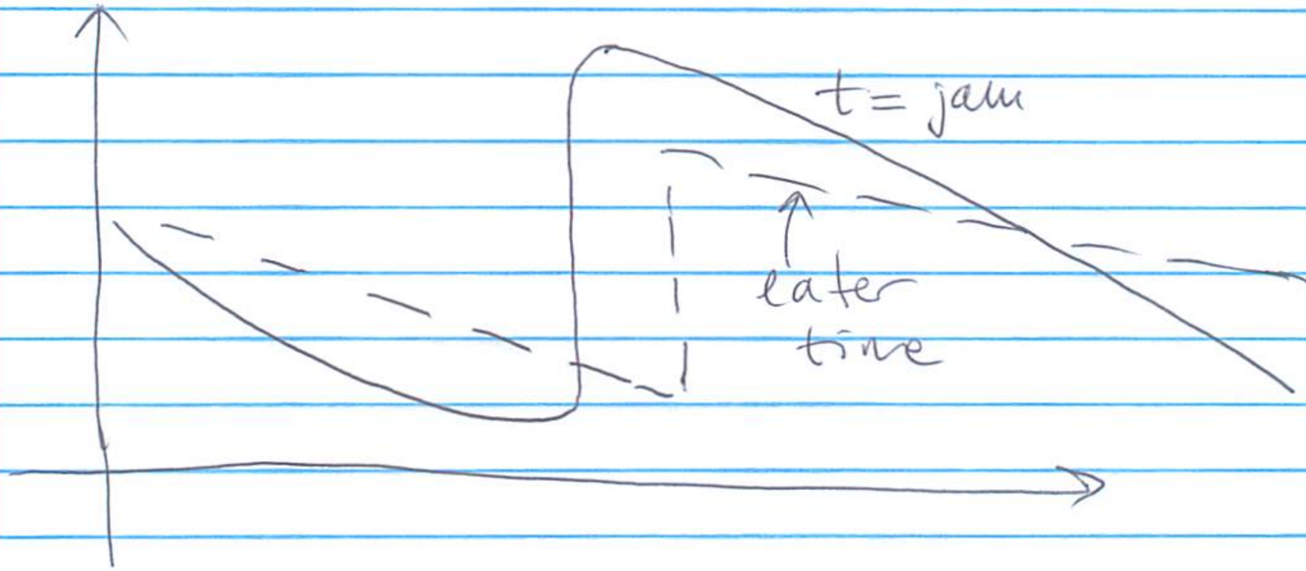
What happens after the discontinuity forms?

One option is to <u>choose</u> one of the multivalued solutions





The discontinuity (jam) moves to the right and the amplitude of the shock diminishes

<u>the shock dissipates</u>

(entropy argument)

Left and right of the shock we can construct a "<u>classical</u>" solution using method of characteristics.

the question is how fast the shock moves to the right?

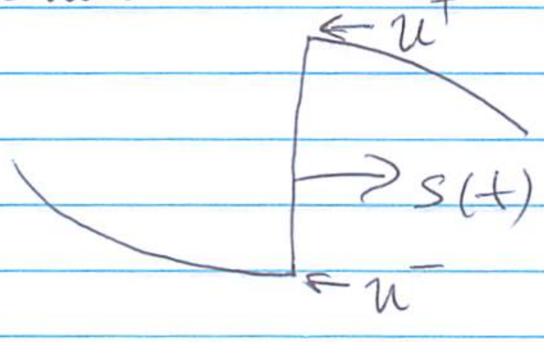
This can <u>only</u> be answered by looking at the weak form of the PDE, i.e. the conservation law itself

See (17) in Strauss PDE book:

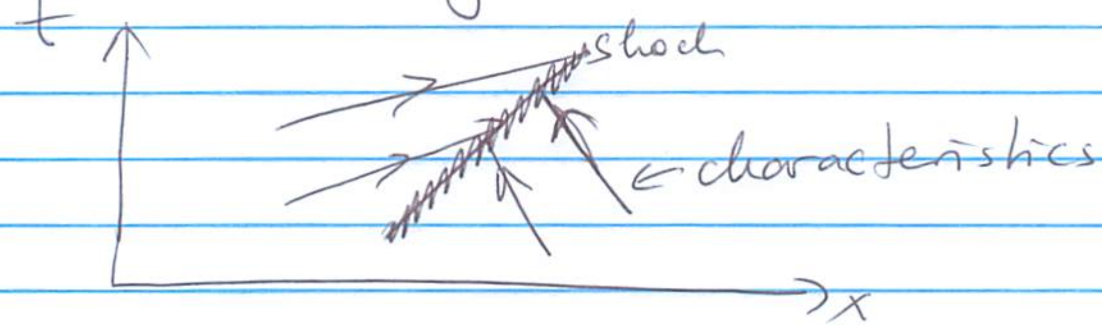
$$s(t) = \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}$$

$\uparrow$ speed of shock

$\uparrow$ Rankine-Hugoniout solution



This is all we need if all of the characteristics go <u>into</u> the shock without emanating from it



$$\varphi'(u^-) \geq \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-} \geq \varphi'(u^+)$$

This is called the <u>Lax condition</u>

To see why the weak form is required and strong is <u>not</u> enough, rewrite PDE as

$w = F(u)$ new flux

$$w_t + G_x = 0 \,, \quad G = \int F'(u)\,\varphi'(u)\,du$$

and now

$$\frac{G^+ - G^-}{F^+ - F^-} \neq \frac{\varphi^+ - \varphi^-}{u^+ - u^-}$$

(11)

Different things happen in other systems.

E.g. for shallow water waves (rivers flowing)

$$\varphi = \frac{u^2}{2}$$

so $\varphi'' > 0$

Now information propagates <u>faster</u> than the water (river), so objects in water are caught in

<u>flash floods</u>

which are now shocks for water waves

Reality { Flash floods come quickly without a warning, but the water rise dissipates slowly afterward

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