PDE Spring 2016

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Lecture 9

MAXIMUM PRINCIPLES & Energy

Let us now try to prove stability and uniqueness for the heat equation IVP.

We will do this in 3 different ways and learn things along the way.

1. Assume we perturb the initial condition by no more than δ:

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \varphi(x) \end{cases} \qquad \begin{aligned} v_t &= k v_{xx} \\ v(x,0) &= \psi(x) \end{aligned}$$

$$\max |\varphi(x) - \psi(x)| \leq \delta$$

Can we bound

$$\max |u(x,t) - v(x,t)| \quad ?$$

Assume here uniqueness, and use our solution formula:

$$w = u - v \Rightarrow$$

$$w_t = k\, w_{xx} \;,\; w(x,0) = \varphi(x) - \psi(x)$$

$$w(x,t) = \int_{-\infty}^{\infty} \underset{\substack{\vee\! / \\ 0}}{G(x-y,t)}\, w(x,0)\, dy$$

$$\Rightarrow$$

$$|w| \leq \int_{-\infty}^{\infty} G(x-y,t)\, |\varphi - \psi|\, dy$$

$$\leq \delta \underbrace{\int G(x-y,t)\, dy}_{\text{unity}} = \delta$$

$$\Rightarrow \boxed{\max |u(x,t) - v(x,t)| \leq \delta}$$

which proves that the perturbation of the solution is no larger than the perturbation of the IC

$\Rightarrow$ IVP <u>is</u> stable!

2 Energy method

Let us now consider a bounded interval BVP — same applies to IVPs of course

$$\begin{cases} u_t - k u_{xx} = f(x,t), \quad 0<x<l \\ u(x,0) = \varphi(x) \\ \text{Dirichlet BC's (example):} \\ u(0,t) = g(t) \quad u(l,t) = h(t) \end{cases}$$

If BVP had two solutions u_1 and u_2, then

$$w = u_1 - u_2$$

would satisfy homogeneous equation.

So if we prove that the only solution to the homogeneous problem is $w \equiv 0$ then we will prove uniqueness since

$$u_1 \equiv u_2$$

Let us do this in two different ways

Multiply PDE by w:

$$w(w_t - k\,w_{xx}) = 0$$

and note that

$$w\,w_t = \frac{\partial}{\partial t}\left(\frac{w^2}{2}\right)$$

Integrate over the interval

$$\int_0^{\ell} \Big[\frac{\partial}{\partial t}\left(\frac{w^2}{2}\right) - \underbrace{k\,w\,w_{xx}}_{\text{integrate by parts}} \Big]\,dx = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_0^{\ell} w^2\,dx = -k\int_0^{\ell} (w_x)^2\,dx + k\Big[w\,w_x\Big]_{x=0}^{\ell}$$

this term is zero for either homogeneous Dirichlet or homogeneous Neumann BCs !

$$\boxed{\begin{aligned} \frac{d}{dt}E &= -k\int (w_x)^2\,dx \le 0 \\ E &= \frac{1}{2}\int w^2\,dx = \underline{\text{energy}} \end{aligned}}$$

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We say that the PDE is dissipative because the energy is "dissipated" $\to$ decays with time.

If the initial condition is

$w(x,0)=0$ then

$\Rightarrow E(t=0)=0 \Rightarrow E(t)=0$

at all times

since $E \geq 0$ by definition

$$\boxed{\begin{array}{l}\text{The only solution of the homogeneous BVP for the heat equation is } w=0 \\ \Rightarrow \text{BVP has a unique solution}\end{array}}$$

We can also prove stability this way as well, since

$0 \leq E(t) \leq E(0)$

$$\int (u-v)^2 dx \leq \int \left(\varphi(x)-\psi(x)\right)^2 dx$$

Note: This is stability in the L_2 norm versus previously the L_∞ norm

Euclidean

abs or max

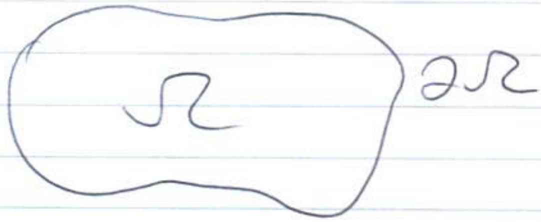
3 Max principles

We now prove uniqueness and stability by another method.

Let's first consider the Laplace equation in 2D

$$u_{xx} + u_{yy} = 0 \quad \text{in } \Omega$$

open, bounded, <u>connected</u>



Either $u = \text{const}$ or u has a maximum or minimum on the boundary $\partial\Omega$

In other words:

Maximum and minimum must be attained on the boundary

This is the maximum/minimum principle for the heat equation

Proof:

Basic idea: (for max)

If there is a maximum inside Ω at point P, then

$$u_{xx} \leq 0 \quad \text{and} \quad u_{yy} \leq 0$$

If $u_{xx} < 0$ or $u_{yy} < 0$ then it cannot be that $u_{xx} + u_{yy} = 0$

The trickier part is the case

$$u_{xx} = 0 = u_{yy}$$

We do this by a "trick" or continuity argument

Define $w = u + \varepsilon(x^2 + y^2)$

$$\nabla^2 w = \nabla^2 u + 4\varepsilon = 4\varepsilon > 0$$

So w is convex and therefore cannot have a maximum inside Ω, must be on boundary $\partial\Omega$ at some point $P \in \partial\Omega$

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$$u = w - \varepsilon (x^2+y^2) < w$$
$$\leq w(P \in \partial\Omega) = u(P) + \varepsilon(x_P^2+y_P^2)$$
$$< u(P) + \varepsilon L^2$$

where L is the radius of the circle enclosing Ω (this is why it has to be bounded)

$$u < u(P) + \varepsilon L^2$$

But as $\varepsilon \to 0$ we get

$$\boxed{u \leq u(P)} \text{ in } \Omega$$

as needed to show.

Now we state the maximum principle for the heat equation in one dimension (same applies in all dimensions).

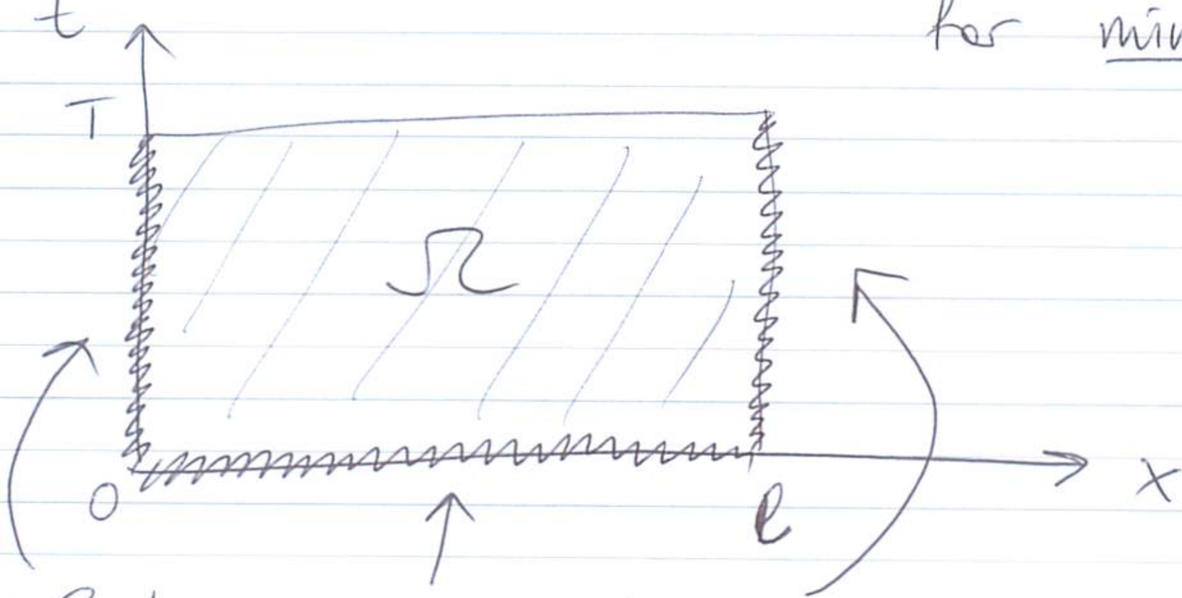
In words: A rod cannot get hotter/colder at any point then the largest/smallest initial or boundary temperature

Maximum principle:

If $u(x,t)$ solves the heat equation in the rectangle

$$\Omega = \{ 0 \le x \le l,\ 0 \le t \le T \}$$

then the maximum value of u is either attained initially or one the lateral sides (same for minimum)



Extremum is achieved on one of these three sides of the rectangle

[This generalizes to other dimensions for $u_t = k \nabla^2 u$

Proof

If there were an extremum inside the domain, then at that point

$$u_x = u_t = 0$$

and $u_{xx} \leq 0$

If $u_{xx} < 0$ and $u_t = 0$ we cannot have $u_t = k u_{xx}$ so this cannot be

The case $u_{xx} = 0$ requires a similar mathematical game as we did for the Laplace equation. Define

$$v(x,t) = u(x,t) + \epsilon x^2$$

and prove things about the strictly convex $v(x,t)$, and then take $\epsilon \to 0$ to show u and v share the extremum. We won't go through this here.

Observe that the maximum principle can be used to prove uniqueness and stability also.

If u_1 and u_2 are two solutions then $w = u_1 - u_2$ has a minimum <u>and</u> maximum of <u>zero</u> since w is zero on the boundaries (both IC and BC are homogeneous).

So <u>$w = 0$ everywhere</u> $\Rightarrow$ uniqueness

Similarly for stability.

$\frac{\text{Max}}{\text{Min}}$ principle says

$$\underbrace{-\max}_{\text{bottom}}(\varphi_1 - \varphi_2) \le u_1 - u_2 \le \underbrace{\max}_{\text{bottom}}(\varphi_1 - \varphi_2)$$

$$\Rightarrow \max |u_1 - u_2| \le \max |\varphi_1 - \varphi_2|$$

which is the same as we proved earlier $\Rightarrow$ <u>stability in L_∞ norm</u>