PDE Spring 2016

Lecture 7

The Wave Equation

We consider the wave equation in 1D in an unbounded domain, i.e., the whole real line.

Treating bounded domains is harder and will be done later.

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad c \neq 0$$

c := speed of wave (sound, etc.)

Statement
The general solution is of form:

$$u(x,t) = f(x+ct) + g(x-ct)$$

for arbitrary functions f and g (continuously differentiable)

= LEFT WAVE + RIGHT WAVE

$\xleftarrow{\quad}$ $f(x+ct)$ $\qquad$ $\xrightarrow{\quad}$ $f(x-ct)$

Let's check this is a solution:

$$\mathcal{L} = \partial_t^2 - \partial_x^2 = (\partial_t + c\partial_x)(\partial_t - c\partial_x)$$

$$\mathcal{L} = \mathcal{L}^+ \mathcal{L}^-$$

Observe

$$\mathcal{L}^+ g(x-ct) =$$

$$(\partial_t + c\partial_x)\, g(x-ct) = 0$$

since we know $g(x-ct)$ solves the advection equation

$$u_t + c u_x = 0$$

similarly,

$$\mathcal{L}^- f(x+ct) = 0$$

So

$$\mathcal{L}[f+g] = \mathcal{L}^+\mathcal{L}^-[f+g]$$

$$= \mathcal{L}^+\mathcal{L}^- g(\underbrace{x-ct}_{y}) =$$

$$= \mathcal{L}^+(-c g'(y) + g'(y)) =$$

$$= \mathcal{L}^+(h(x-ct)) = 0$$

Now let us derive this solution and therefore show that it is general, i.e., <u>every</u> solution is of form

$$u = f(x+ct) + g(x-ct)$$

Denote $\mathcal{L}^{+}u = v = u_t + c u_x$

$$\Rightarrow \mathcal{L}u = \mathcal{L}^{-}v = 0$$

$$\mathcal{L}u \iff \begin{cases} \mathcal{L}^{+}u = v \\ \mathcal{L}^{-}v = 0 \end{cases}$$

i.e. we have converted the second-order PDE into a system of first-order advection equations, which we know how to solve

Note: <u>This</u> is the same as for ODEs

$$u''(t) = f(u,t) \iff$$

$$\begin{cases} u' = v \\ v' = f(u,t) \end{cases}$$

$$\bar{\mathcal{L}} v = v_t - c v_x = 0 \Rightarrow$$
$$v = h(x+ct) \Rightarrow$$
$$u_t + c u_x = h(x+ct)$$

Try to show that this implies
$$u = f(x+ct) + g(x-ct)$$

But in fact there is a simpler and more general way here:

Use <u>characteristic coordinates</u> to transform PDE into the canonical form of hyperbolic PDEs $u_{\xi\eta} = 0$

xi eta

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

$$\begin{cases} \partial_x = \partial_\xi + \partial_\eta \\ \partial_t = c\partial_\xi + c\,\partial_\eta \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{L}^- = -2c\,\partial_\eta \\ \mathcal{L}^+ = 2c\,\partial_\xi \end{cases}$$

$$\mathcal{L} u = -4c^2\, \partial_\eta \partial_\xi u = 0$$

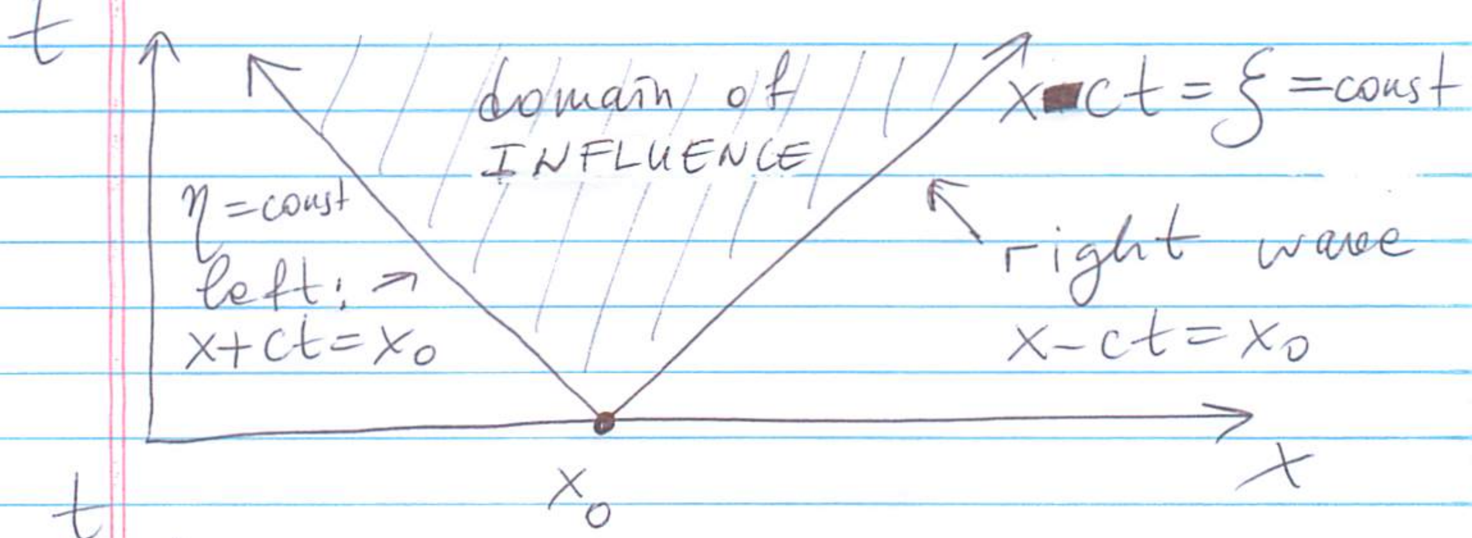
$$\Rightarrow \quad \boxed{\partial_{\eta\xi} u = 0} = \partial_\xi(\partial_\eta u)$$

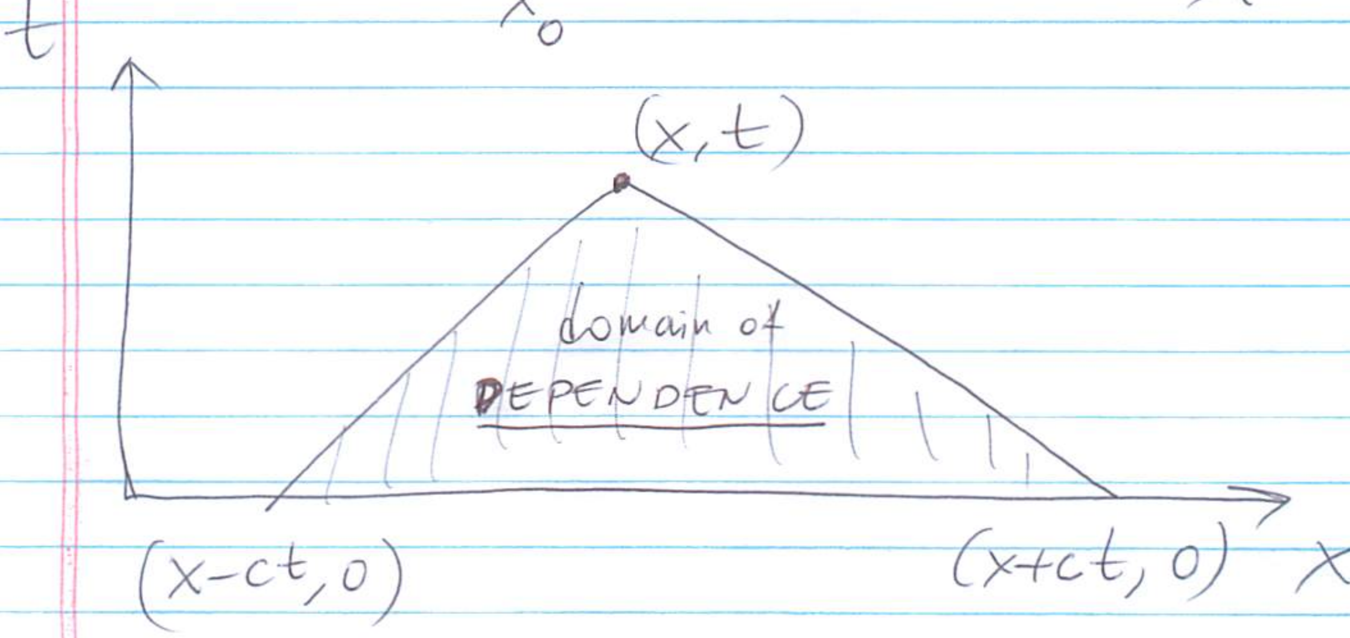
This is the first sort of equation we considered in this class, and we know this.

$$\begin{cases} \partial_\eta u = v \Rightarrow \partial_\xi v = 0 \Rightarrow \\ v = f(\eta) = \partial_\eta u \Rightarrow \\ u = F(\eta) + G(\xi), \quad F' = f \end{cases}$$

$$u = f(\eta) + g(\xi)$$

is the *general* solution, as we claimed.





Information propagates at a maximum speed of c

(key property of hyperbolic PDEs)

We know $u = f(x+ct) + g(x-ct)$
but what are f and g?

Since we have two unknown functions we need <u>two ICs</u> (since there are no boundaries we cannot specify a BC here)

$$\begin{cases} u(x,0) = \varphi(x) \quad \text{phi} \\ u_t(x,0) = \psi(x) \\ \quad \text{psi} \end{cases}$$

$$\begin{cases} u(x,0) = f(x) + g(x) = \varphi(x) \\ u_t(x,0) = c f'(x) - c g'(x) = \psi(x) \end{cases}$$

Differentiate first equation to get the system

$$\begin{cases} f' + g' = \varphi' \\ f' - g' = \psi/c \end{cases}$$

$$\Rightarrow \begin{cases} f' = \frac{1}{2}(\varphi' + \psi/c) \\ g' = \frac{1}{2}(\varphi' - \psi/c) \end{cases}$$

$$\Rightarrow f = \frac{1}{2}\int (\varphi' + \psi/c)\, dx$$

$$= \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\, ds + A$$

integration constant

Similarly

$$g = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\, ds + B$$

$$\Rightarrow f + g = \varphi + (A+B) \Rightarrow$$

$$A + B = 0$$

(we got an extra degree of freedom here because we differentiated $f + g = \varphi$ first)

$$u = f(x+ct) + g(x-ct)$$

$$\boxed{u = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds}$$

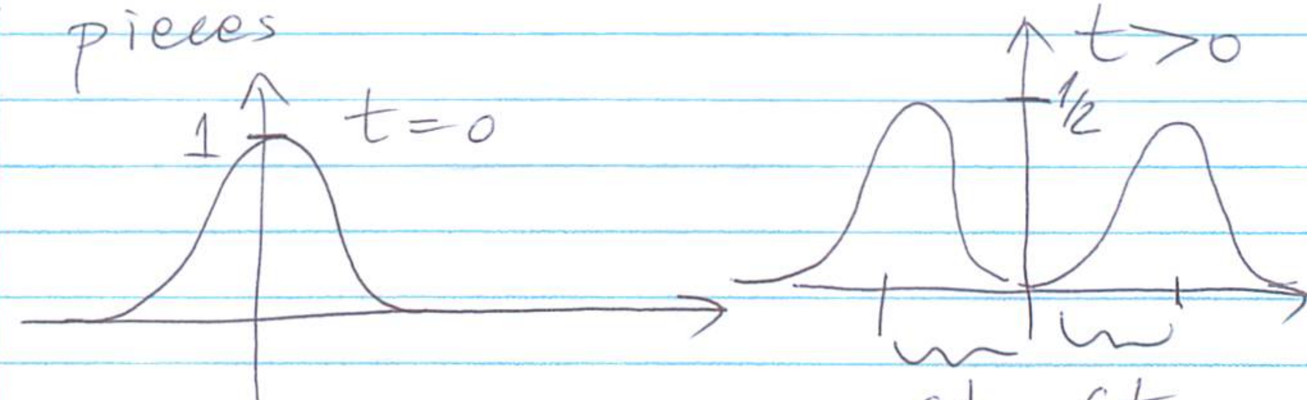
d'Alambert's formula

We have now shown <u>existence</u> and <u>uniqueness</u> of the IVP (<u>Cauchy</u> probem for the wave equation)

Observe that d'Alambert's formula conforms to the picture about the <u>domain of dependence</u>:

φ or ψ outside the domain do <u>not</u> affect the solution

<u>Example</u>: If $\psi = 0$ then the initial profile $\varphi = u(x,0)$ splits in <u>half</u> into two pieces



How about stability?
Is the Cauchy problem well-posed?
YES

Let u_1 and u_2 be two solutions for different ICs

$u = u_1 - u_2$ also a solution

$$|u(x,t)| \leq \max|\varphi|$$

$$+ \frac{1}{2c} \cdot \max|\psi| \cdot \underbrace{2ct}_{\text{width of interval}}$$

$\Rightarrow$

$$\max|u_1 - u_2| < \max|\varphi_1 - \varphi_2|$$

$$+ T \max|\psi_1 - \psi_2|$$

where $0 \leq t \leq T$

This means that for finite time small perturbations of the ICs induce small perturbations in solution.