PDE Spring 2016

A. Donev

Lecture 5: Advection Equations

Recall basic first-order linear advection equation

$$u_t + c\,u_x = 0$$

Here we will consider a generalization

$$a\,u_x + b\,u_y = f$$

and use y instead of t to follow 4.1 in Essential PDE book

We will solve this (trivial) PDE in two different ways, each useful later for second-order PDEs.

Change of dependent variables
or change of coordinates

$$a u_x + b u_y = 0$$

$$\begin{cases} x' = ax + by \\ y' = bx - ay \end{cases}$$ new coordinates

$$u(x,y) \equiv u(x',y')$$

Use chain rule

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} =$$

$$= a u_{x'} + b u_{y'}$$

Or in operator notation

$$\partial_x = a\,\partial_{x'} + b\,\partial_{y'}$$

(More generally use Jacobian of transformation
review!)

Similarly

$$\partial_y = b\,\partial_{x'} - a\,\partial_{y'}$$

So

$$a u_x + b u_y \underset{\uparrow}{=} (a^2+b^2)\, u_{x'}$$

do algebra on your own!

Since $a^2+b^2 > 0$ unless trivial equation

$$u_{x'} = 0 \quad \Rightarrow$$

$$u = f(y') = f(bx - ay)$$

where f is an arbitrary differentiable function

$$\boxed{u = f(bx-ay)} =$$

$$= f(\underbrace{\text{characteristic coordinate}}_{y'})$$

Apply this to the <u>advection equation</u>

$$u_t + c\,u_x = 0$$

$$a \equiv c \,, \quad b \equiv 1$$

$$u = f(x - ct)$$

where

$f(x) = u(x,0)$ is the <u>initial condition</u>

But what if the advection velocity (speed) depends on position?

$$u_t + c(x,t)\,u_x = 0$$

(still linear but not constant coeff)

In this case it is better to take a similar but distinct approach

Method of characteristics

$$p u_x + q u_y = f$$

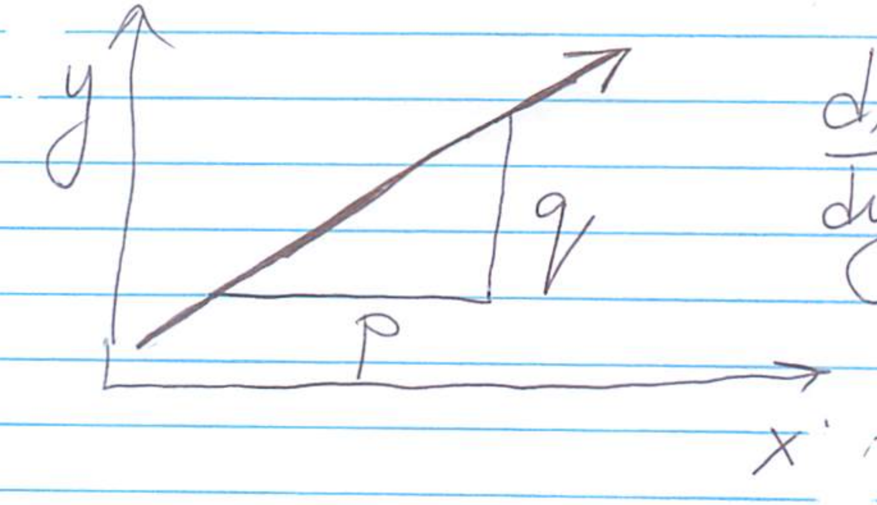
Consider tracking the solution along a curve in the plane, parametrized by some parameter

$$w = u(x(t), y(t))$$

$$\frac{dw}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$$

Now, let's choose

$$\frac{dx}{dt} = p \qquad \frac{dy}{dt} = q$$



$$\frac{dx}{dy} = \frac{p}{q}$$

6

$$\frac{dw}{dt} = p u_x + q u_y \underset{\uparrow}{=} f$$

PDE

The choice of t is irrelevant since we can eliminate dt and instead of

$$\frac{dx}{dt} = p \; ; \; \frac{dy}{dt} = q \; ; \; \frac{dw}{dt} = f$$

write

$$\boxed{\frac{dx}{p} = \frac{dy}{q} = \frac{dw}{f}}$$

which describes the solution along a characteristic line, which is the solution of

$$\frac{dx}{p} = \frac{dy}{q} \quad \Big\} \text{ system of ODEs}$$

The solution along characteristic solves another ODE

$$\frac{dw}{dt} = f(x, t)$$

We have therefore reduced the PDE to a system of ODEs

Geometric interpretation

$$p u_x + q u_y = \frac{\partial u}{\partial \vec{s}}$$

Directional derivative along
$\vec{s} = (p, q)$

So PDE says

$$\frac{\partial u}{\partial \vec{s}} = 0 \implies u = \text{const}$$

but const is different for each characteristic

Take home:

Solution is constant along characteristics
or
solution is transported along characteristics

Examples :

Method of characteristics works for non-constant coefficients, inhomogeneous equations, and even nonlinear equations like Burgers.

ex. 1 Solve

$$p u_x + u_y = u$$

where $p = \text{const.}$

$$\frac{dx}{p} = \frac{dy}{1} = \frac{du}{u}$$

$$\Rightarrow \frac{dx}{dy} = p \Rightarrow x = py + k$$

$$\frac{du}{dy} = u \Rightarrow u = Ae^{y}$$

The value of k and A will be different for each characteristic. Let's use k to parametrize the characteristics

$$x = py + k$$

$$u = A(k)e^y = A(x - py)e^y$$

$$\boxed{u(x,y) = A(x - py)e^y}$$

where A is determined from the initial / boundary conditions

E.g.

$$u(x, y=0) = A(x) = IC$$

<u>Example 2</u>

Solve $u_t + 2t u_x = 0$

$$p \equiv 2t \qquad q \equiv 1 \qquad y \equiv t$$

$$\frac{dx}{dt} = 2t \quad \text{or} \quad x = t^2 + c$$

$$\frac{dw}{dt} = 0 \implies w = \text{const}$$

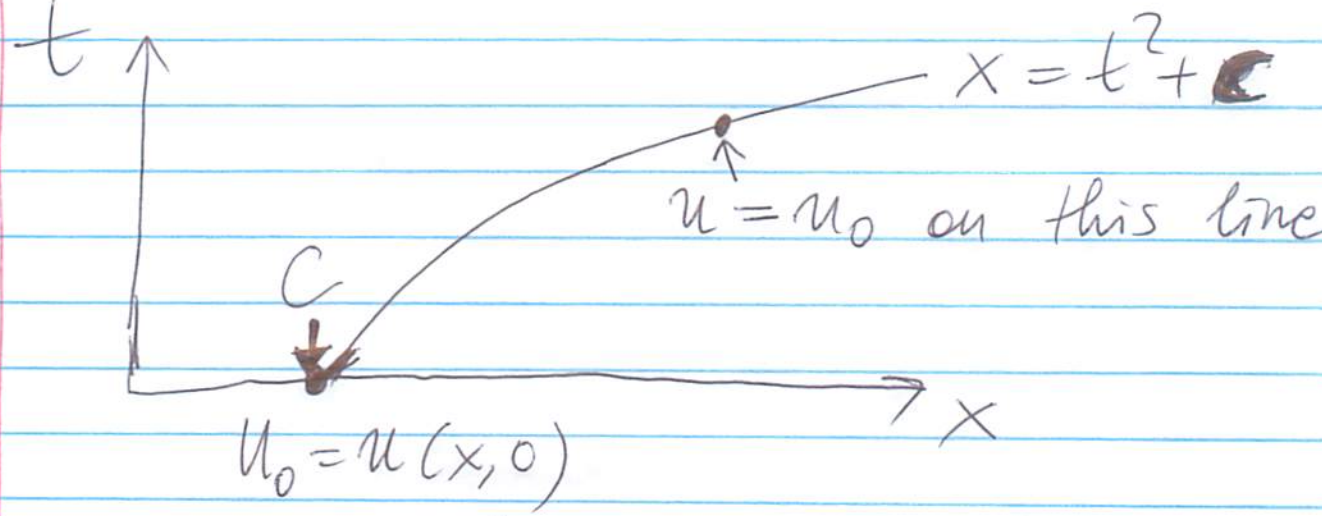
$$u(x(t), t) = \text{const}$$

$$u(t^2 + c, t) = u(x(0), 0)$$

$$u(t^2 + c, t) = u(c, 0)$$

$$c = x - t^2$$

$$u(x,t) = u(x - t^2, 0)$$



So to find $u(x,t)$ you need to <u>trace the characteristics</u> back to the initial or boundary condition and just read off the value!

= Geometry

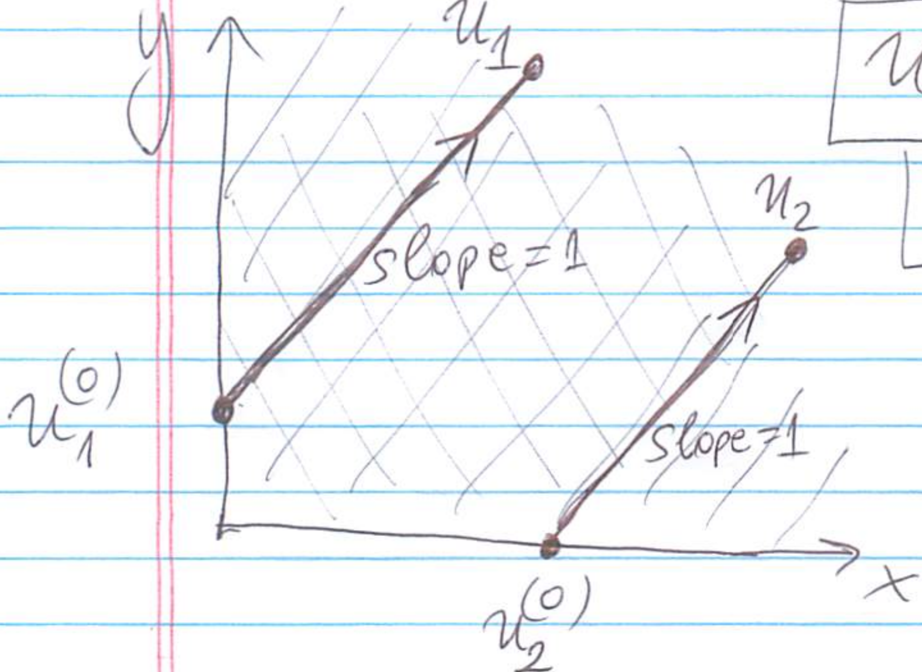
In more complicated cases the formulas may be more complicated but the picture is always the same and instead of algebra it is better to draw and do geometry instead

Practice

$$\begin{cases} \text{Solve} & u_x + y u_y = 0 \\ \text{Solution} & u(x,y) = f(e^{-x} y) \end{cases}$$

Boundary conditions and characteristics

1 Quarter-plane problem



$$\boxed{u_x + u_y = 0}$$

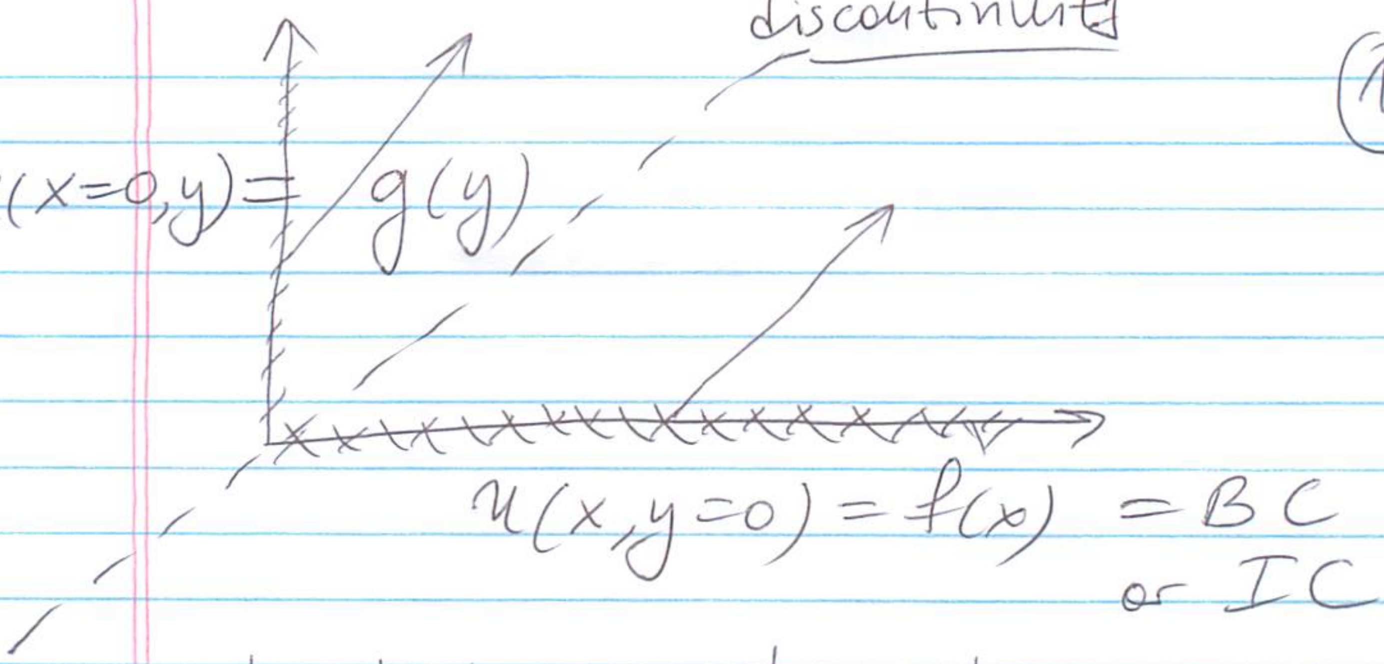
$$\boxed{x > 0, \; y > 0}$$

$x - y = \text{const}$

characteristics have slope $= 1$

$$\begin{cases} u_1 = u_2^{(0)} \\ u_2 = u_1^{(0)} \end{cases}$$

So ~~to~~ determine the solution along <u>every</u> characteristic in the domain (first quadrant) we need to know u along $x=0$ <u>and</u> $y=0$



Note that we do <u>not</u> require

$$f(0) = g(0)$$

If $f(0) \neq g(0)$ then the solution will be <u>discontinuous</u> along the characteristic $x = y$

(2) Now how about

$$u_x - u_y = 0 \quad ?$$

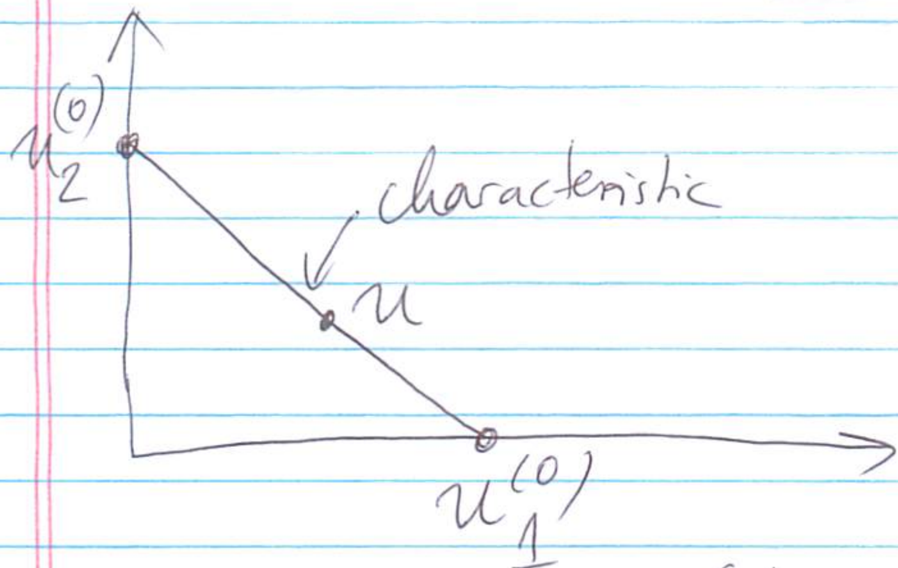
$$\Rightarrow \text{slope} = -1$$

Can we now specify

$$u(x=0,y) = g(y) \quad \underline{\text{and}}$$

$$u(x,y=0) = f(x) \quad ?$$

the answer is **no**



$$u = u_1^{(0)} = u_2^{(0)}$$

so we cannot specify u independently on both boundaries

If we assign a direction to time and thus to the characteristics, the statement is

Boundary conditions should only be imposed on boundaries where the characteristics are directed into the domain (flow of information with time)