# PDE Spring 2016

## Lecture 4

~~[illegible]~~

## Conservation Laws

Reading: EPDE: 3.2, 3.2.1
APDE: Section 1.2, Ex. 1.15
1.3, 1.7

We consider the transport of a conserved quantity in 1D

Examples:

→ Traffic flow

→ Polutant / smoke in air flow

→ Heat (energy) flow through metal

Traffic Lane on Highway

$u(x,t)$ = density of cars / molecules

# Quantity = $u(x,t)\,dx$

Flux $\varphi(x,t)$ = amount of quantity passing through $x$

Positive means to the right ②

E.g. number of cars passing through intersection / mile marker per second

$f(x,t)$ = source / sink of quantity

E.g: Traffic coming onto highway from entrance (+) or leaving from exit (−)

Fundamental <u>conservation law</u>

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \varphi(a,t) - \varphi(b,t) + \int_a^b f(x,t)\,dx$$

This is called a "weak form" of a conservation PDE. Under some assumptions, it can be converted into a traditional PDE, which is called the "strong form" and is the focus of this course.

Recall from ODEs that

$$x'(t) = f(x,t)$$ is equivalent to

~~[illegible]~~ integral equation

$$x(t) = x(0) + \int_0^t f(x(t),t)\,dt$$

which is in fact more general than ODE (e.g. stochastic ODEs)

If function is sufficiently smooth, i.e. sufficiently continuously differentiable, we can convert conservation law to a PDE.

Lots of PDEs in practice come from conservation laws!

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t\,dx$$

**if** $u$ has continuous first partial derivatives

Similarly, if $\varphi$ has continuous first partials, then fund. th. of calculus says

$$\varphi(a,t) - \varphi(b,t) = -\int_a^b \varphi_x\,dx$$

Conservation law becomes:

$$\int_a^b \left[ u_t + \varphi_x - f \right] dx = 0$$

for all $a$ and $b$

Since integrand is continuous (crucial!) and $a$ and $b$ are arbitrary, the above implies integrand identically vanishes

$$\boxed{u_t + \varphi_x = f}$$

conservation law PDE

To turn this into a PDE we need a <u>constitutive law</u>

$$\varphi \equiv \varphi(u, x, t)$$

which is problem-specific.

For traffic flow

$$\varphi = u\,c$$

where $c$ is speed of cars

If $c$ is constant (boring!) then we get advection equation

$$u_t + c\,u_x = f$$

But in the real world $c$ depends on $u$ (and also $x, t$ due to road/weather conditions)

So to get an actual PDE we need to do

<u>mathematical modeling</u>

E.g.

$$c = c_{max}\left(1 - \frac{u}{u_{max}}\right)$$

where $u_{max}$ is the "jamming" density of cars

$\Longrightarrow$ (derive as practice)

$$u_t + \underbrace{c_{max}\left(1 - 2\frac{u}{u_{max}}\right)}_{\substack{\text{Speed of traffic "waves"} \\ \text{or speed of propagation} \\ \text{of information through} \\ \text{the highway}}} u_x = 0$$

The above is a nonlinear advection equation similar to Burgers equation

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We will not study nonlinear advection in detail but see text books for advanced students.

Here, as practice, let's show that the *implicit* solution

$$\left\{\begin{array}{l} u = F(\underbrace{x - c(u)t}_{y}) \\ \text{solves} \\ u_t + c(u)u_x = 0 \end{array}\right.$$

$$u_t = F'(y)\left[-c(u) - c'(u)u_t\, t\right]$$

$$u_x = F'(y)\left[1 - c'(u)u_x\, t\right]$$

$$c\,u_x = F'(y)\left[c - c'c\,u_x t\right]$$

$$u_t + cu_x = -c'F'(y)t\left[u_t + cu_x\right]$$

$$\Rightarrow \text{ if } c'F' \neq 0 \quad u_t + cu_x = 0$$

## Diffusion
as a conservation law

We all know that heat "flows" from hot to cold.

In general, the higher the gradient of temperature the larger the flow.

So it seems reasonable to postulate

$$\varphi = -k u_x$$

where $k > 0$ is a diffusion constant

$\rightarrow$ flux is "down the gradient"

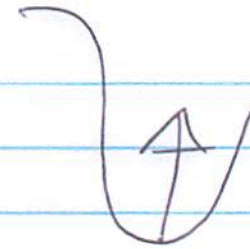
$$\boxed{u_t + (k u_x)_x = 0}$$

Heat equation as a conservation law

Here $k = k(x)$ or even
$k = k(u, x, t) > 0$ works

If $u_{xx} > 0$

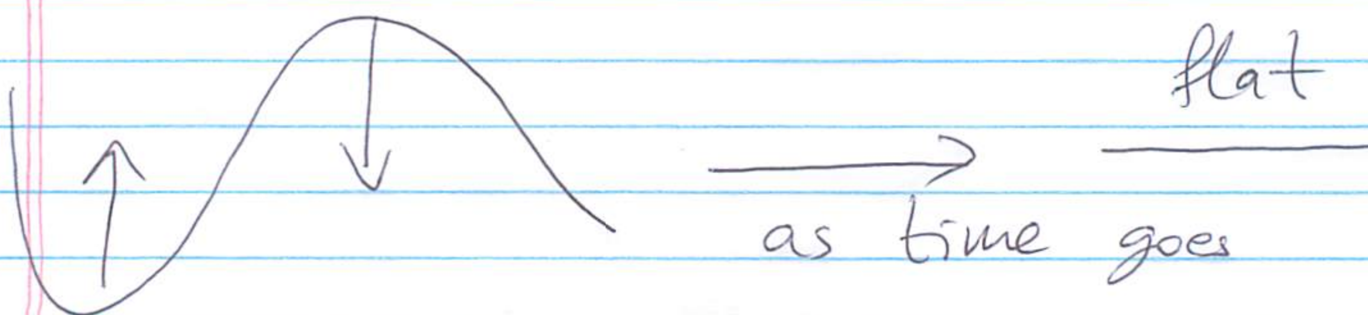
$\Rightarrow u_t > 0$



convex

If $u_{xx} < 0$

concave

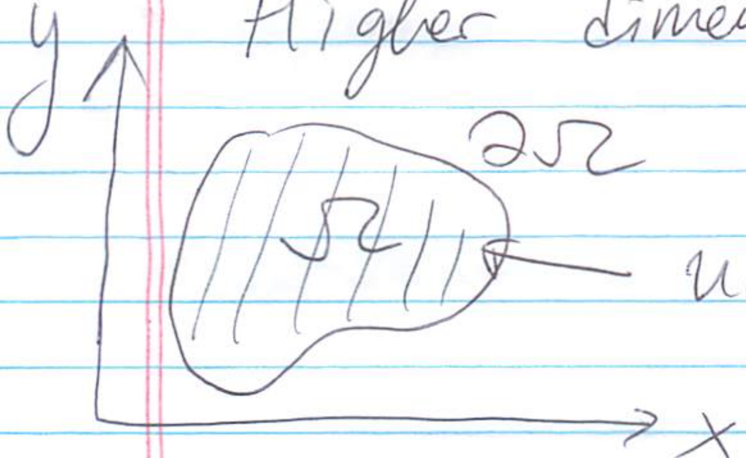



Diffusion "flattens" or "smears" the solution

It makes it <u>smoother</u> with time

There is a deep connection between diffusion and randomness or <u>random walks</u>, see 1.4 in Applied PDE book if interested in details

# Higher dimensions




$u(x,y) : (x,y) \in \Omega$

$\partial\Omega$ = boundary of $\Omega$

$$\frac{d}{dt}\int_V u \, dV = \int_{\partial V} (\vec{\varphi}\cdot\vec{n})\, dS + \int_V f \, dV$$

where $V \subseteq \Omega$ is an arbitrary subvolume

Here flux $\vec{\varphi}$ is a vector and $\vec{\varphi}\cdot\vec{n}$ is the normal flux through the boundary of a subvolume

Since $V$ is fixed with time

$$\frac{d}{dt}\int_V u\,dV = \int_V \frac{\partial u}{\partial t}\,dV$$

Also recall divergence theorem

Green & Gauss generalize fund. theorem of calc.

$$\int_\Omega \nabla \varphi \, dV = \int_{\partial\Omega} \varphi \vec{n}\, ds$$

$$\int_\Omega (\vec{\nabla}\cdot\vec{\varphi})\, dV = \int_{\partial\Omega} \vec{\varphi}\cdot\vec{n}\, ds$$

$$\Rightarrow \int_V \left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{\varphi} - f\right) dV = 0$$

for all $V \subseteq \Omega$

(weak form)

$\Downarrow$ certain assumptions

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{\varphi} = f}$$

Strong form

Advection in higher dimensions $\quad c \to \vec{c}$ (vector)

$\vec{c}\,(x,y) \to$ vector field in general

$$\vec{\varphi} = u\,\vec{c}$$

$$\Rightarrow \quad u_t + \vec{\nabla}\cdot(u\,\vec{c}) = 0$$

if $\vec{c}$ is a constant vector

$$u + \vec{c}\cdot(\vec{\nabla} u) = 0$$

Diffusion in higher dimensions

$$\vec{\varphi} = -k\,\vec{\nabla} u \quad , \quad k>0$$

$$u_t + \vec{\nabla}\cdot(-k\,\vec{\nabla} u) = 0$$

If $k$ constant

$$u_t = k\,\vec{\nabla}\cdot(\vec{\nabla} u) = k\,\nabla^2 u$$

Practice: Show $\vec{\nabla}\cdot\vec{\nabla} \equiv \nabla^2$

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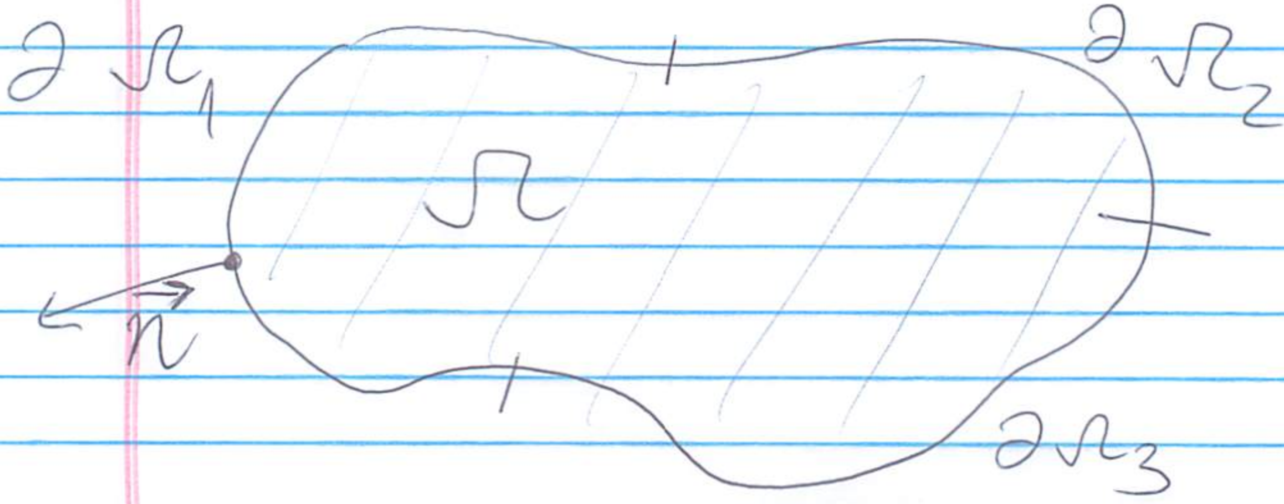
# Boundary conditions in higher dimensions

**Dirichlet BCs** specify values of the solution on the boundary of the spatial $(x,y,z)$ domain

**Neumann BCs** specify values of the flux on the boundary.
E.g.

$$\text{flux} = 0$$

means insulating / impenetrable boundary




$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$$

boundary of domain $\Omega \subseteq \mathbb{R}^2$

Example BCs:

(a) Dirichlet on $\partial\Omega_1$:

$$u(\vec{r}=(x,y) \in \partial\Omega_1) = 0$$

or we write

$$\boxed{u(\partial\Omega_1) = 0}$$

(b) Neumann on $\partial\Omega_2$:

$$\frac{\partial u}{\partial n} = (\vec{\nabla} u) \cdot \vec{n}$$

Normal derivative

gradient

normal to boundary

dot product

$$\boxed{\frac{\partial u}{\partial n}(\partial\Omega_2) = 0} = u_x n_x + u_y n_y$$

$$\vec{\nabla} u = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

(c) No BC can be specified on $\partial\Omega_3$

Which type of BC is allowed on which <u>part</u> of boundary depends on equation

<u>Examples</u>

second order

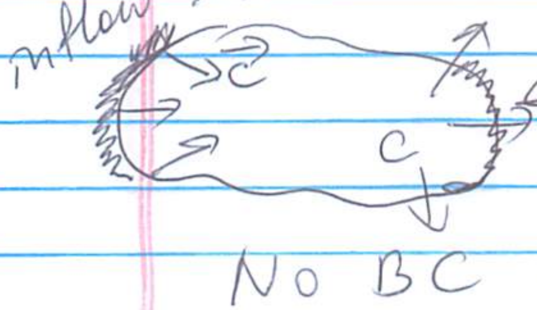
1. For diffusion/heat eq. or Poisson/Laplace

$u_t = k\nabla^2 u$ in $\Omega$

<u>every</u> part of the boundary must have a (Dirichlet/Neumann) BCs

2. For advection equation, <u>inflow</u> boundaries require a Dirichlet condition

inflow $\equiv$ $\vec{c}\cdot\vec{n} < 0$




outflow $\equiv$ $\vec{c}\cdot\vec{n} \geq 0$

No BC on outflow boundaries