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Lecture 3

PDE Classification:

Linear PDEs

Let's introduce the concept of an Operator: A "function of a function" or a function with an infinitely many arguments, or an infinite-dimensional generalization of a function.

\[ L(u) \quad \text{or} \quad L[u(t)] \]

caligraphic \( L \)

For linear PDEs we use:

\[ L(u) = Lu \]

Looks like matrix-vector multiplication.
Examples

\[ \nabla u = \nabla^2 u \]
\[ \nabla u = \nabla \cdot \nabla u \] - Laplacian
\[ \nabla u = \nabla u \] - Divergence
\[ \nabla u = \nabla u \] - Gradient
\[ \nabla u = u_t - k u_{xx} \]

This is just shorthand notation.

If \( \nabla \) is the Laplacian operator, the heat equation can be written as:

\[ u_t = \nabla u \]

Other shorthand notation:

\[
\begin{align*}
\nabla = \partial_{xx} & \Rightarrow \nabla u = u_{xx} \\
\nabla = \nabla^2 & \Rightarrow \nabla u = \nabla^2 u \\
\nabla = \partial_t - k \partial_{xx} & \Rightarrow \nabla u = u_t - k u_{xx} \\
\n\nabla = \partial_{tt} - c^2 \partial_{xx} & \Rightarrow \nabla u = \partial_{tt} - c^2 \partial_{xx} \quad \text{etc.}
\end{align*}
\]

Can be hard to read or ambiguous, but it's convenient.
(3) $Y$ is a linear operator if

- **Scaling**
  \[ Y(dx) = dX, \quad d \in \mathbb{R} \]

- **Linearity**
  \[ Y(x+u) = Y(x) + Y(u) \]

- **Superposition**
  E.g. \[ x = 2x \] follows from basic calculus.

But \[ X(x) = uu_x \] is not linear:

\[ Y(dx) = (dx)(dx)_x = d^2u \]

\[ \Rightarrow Y(dx) = d^2X(x) \]

So this is of quadratic order and not linear.

It means Burger's equation is not linear.

Standard Dirichlet and Neumann \( \underline{\text{nonlinear}} \) BCs are also \( \underline{\text{nonlinear}} \).

E.g.

\[ u(x=0,t) = f \Rightarrow (dxu)(x=0,t) = dx \]

\[ u_x(x=0,t) = f \Rightarrow (dxu)_x(x=0,t) = dx_u \]

and similarly for addition.
Linear PDEs

Definition: A PDE is linear if it has the form

\[ \nu = f(x,t) \]

with \( \nu \) a linear operator, and the BCs are linear.

If \( f = 0 = \nu \) we call the PDE homogeneous, otherwise inhomogeneous. To be homogeneous also all boundary and initial data must be zero.

E.g. \( \begin{cases} \nu = 0 = U_{xx} + U_{yy} \\
U(x=0,y) = 0 \\
U(x=1,y) = 0 \quad \text{All must be zero} \\
U_x(x,y=0) = 0 \\
U_x(x,y=1) = 0 
\end{cases} \)

Abstract notations:

\[ \nu = f \]
\[ \nu = g \quad \text{or just } \quad \nu = f \quad \text{for short} \]

Linear operators

Both PDE equation and BCs & ICs
\[ L = \text{eq} + \text{BCs} + \text{ICs} \]

The most important property of linear PDEs is the

**Superposition Principle**

1. If \( u_1 \) and \( u_2 \) are two solutions of a linear PDE \( \Delta u = 0 \) (homogeneous), then so is any linear combination

\[ u = \lambda u_1 + \beta u_2 \quad , \quad \lambda, \beta \in \mathbb{C} \]

Since

\[ \Delta u = \Delta (\lambda u_1) + \Delta (\beta u_2) = \]

\[ = \lambda (\Delta u_1) + \beta (\Delta u_2) = 0 \]

2. If \( \tilde{u} \) is a "particular" solution of the inhomogeneous

\[ \Delta u = f \]

then if \( \Delta \tilde{u} = 0 \),

\[ u = \tilde{u} + \Delta \tilde{u} \]

is also a solution of \( \Delta u = \Delta (\tilde{u} + \Delta \tilde{u}) = \]

\[ = \Delta \tilde{u} + \Delta \Delta \tilde{u} = \Delta \tilde{u} = f \]
A consequence of \#1 + \#2 is:

\[
\begin{align*}
\text{If } \mathbf{u} = 0 \text{ has a unique solution } \mathbf{X}
\text{, then } \mathbf{u} = 0 \text{ is the only trivial solution of } \mathbf{Xu} = 0.
\end{align*}
\]

Proof:
Suppose there are two solutions \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \):

\[
\begin{align*}
\mathbf{Xu}_1 &= 0 \\
\mathbf{Xu}_2 &= 0
\end{align*}
\]

\[
\mathbf{X}(\mathbf{u}_1 - \mathbf{u}_2) = 0.
\]

(1) If \( \mathbf{Xu} = 0 \) has only \( \mathbf{u} = 0 \) as the solution, then \( \mathbf{u}_1 = \mathbf{u}_2 \) so there is only one solution.

(2) If \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) were two distinct solutions (so not unique) then \( \mathbf{u}_1 - \mathbf{u}_2 \neq 0 \) and \( \mathbf{Xu} = 0 \) has a non-trivial solution.

Conclusion:

Linear BVPs have zero, one, or infinitely many solutions.

Compare to \( \mathbf{Ax} = 0 \) in linear algebra class.
Well-posedness

A BVP that has a unique solution which varies continuously with the initial and boundary data is well-posed. All other BVPs are ill-posed.

well-posed = existence + uniqueness + stability

Here is what stability means. Take

\[ X_n = f \]

and perturb the "data" \( f \):

\[ X (n + \delta n) = f + \delta f \]

perturbation of the solution

Stability implies that

If \( \delta f \) is small, so is \( \delta n \)

Small perturbations of the data lead to small perturbations of the solution.

\[ \| \delta u \| \leq C \| \delta f \| \]
Here \( \| f \| \) denotes the norm of the function \( f \).
An example is the \( L_2 \) norm
\[
\| f \|_2 = \sqrt{\int \| f \|^2 \, dx}
\]
which is like the Euclidean vector norm
\[
\| x \|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}
\]
There are other norms and the choice of norm matters — we will come back to this.

**Example**: Heat equation
\[
\begin{cases}
U_t = U_{xx} & \text{is well-posed} \\
U(x, 0) = f(x) & \text{(we will prove this later on)}
\end{cases}
\]
But "backward" heat eq.
\[
\begin{cases}
U_t = -U_{xx} & \text{is NOT well-posed} \\
U(x, 0) = f(x)
\end{cases}
\]
Concrete notation (less confusion)

\[
x \in \mathbb{R} \\
t \in \mathbb{R}^+
\]

\[
\begin{align*}
U_t &= -U_{xx} \\
U(x, 0) &= f(x)
\end{align*}
\]

Abstract notation (more elegant and general)

\[
\begin{align*}
\Delta U &= g \\
\Delta U &= \{ u_t + U_{xx} \} \\
\Delta u &= f(x) \\
g &= \frac{1}{2} f(x) \\
\text{not the same } f \\
\text{so I used } g \text{ for one}
\end{align*}
\]

Subtract first from second to get

\[
\begin{align*}
(\delta u)_t &= -(\delta u)_{xx} \\
(\delta u)(x, 0) &= \delta f(x)
\end{align*}
\]

One example is

\[
\delta u = U_n = A \cos(mx) e^{-nt}
\]

(Homework 1 shows this is a solution)

\[
\Rightarrow \delta f = f_n = U_n(x, t=0) = A \cos(mx)
\]

\[
\| \delta u \| = e^{nt} \| A \cos(mx) \|= e^{nt} \delta f
\]
\[ \psi + \mathbf{1} \mathbf{8} \mathbf{w} \mathbf{1} \mathbf{1} = e^{n^2 t + \mathbf{1} \mathbf{1} \mathbf{8} t + \mathbf{1} \mathbf{1}} \]

\[ \mathbf{1} \mathbf{8} \mathbf{w} \mathbf{1} \mathbf{1}(t) = C_n(t) \]

\[ C_n(t) = e^{n^2 t} \]

but \[ C_n \to \infty \] for any \( t > 0 \)

if \( n \to \infty \).

So we can make the solution differ by an arbitrarily large amount at any positive time.

The PDE has no stability and is ill-posed. The solution exists and is unique but not stable.

**Physics**: The direction of time is set and time cannot be reversed (second law of thermodynamics).

One can add an inhomogeneous term:

\[ \begin{cases} u_t = u_{xx} + g(x) \\ u(x, 0) = f(x) \end{cases} \]

This is well-posed.

\[ \begin{aligned} \Delta u &= h \\ \Delta u &= \{ u(x, 0) \} \\ h &= \{ g(x), f(x) \} \end{aligned} \]
Continuum superposition \[ y_n = 0 \]

Let \( u(x, t; \eta) \) be a solution, where \( \eta \) = parameter.

E.g. \( u(x, t; \eta) = e^{-\eta^2 t} \cos(\eta x) \) for the heat equation.

Then, coefficient or weight

\[ w = \int \phi(\eta) u(x, t; \eta) \, d\eta \]

\( \eta \in \mathcal{E} \) greek "eta"

is also a solution for sufficiently smooth \( \phi(\eta) \)

**Proof:**

\[ \frac{\partial u}{\partial t} = \frac{1}{\nu} \int_{\eta} \phi(\eta) \frac{\partial u}{\partial \eta} \, d\eta \]

Can move inside if integrand smooth

\[ = \int_{\eta} \phi(\eta) \frac{\partial u}{\partial \eta} \, d\eta = \int_{\eta} \phi(\eta) \phi(\eta) \, d\eta = 0 \]

Complex numbers
Example from Lecture 2

\[ u(x, t; s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-s)^2}{4t}} \]

or origin

is a solution of heat equation on \((\mathbb{R}, \mathbb{R}^+))\) for any \(s\).

\[ \Rightarrow \text{So is the continuous superposition:} \]

\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s)e^{-\frac{(x-s)^2}{4kt}} \, ds \]

Algebra is now easy!

We will show later that by a suitable choice of \(f\) we can solve the IVP for any initial condition. In fact (sneak preview / spoiler)

\[ f(x) = u(x, t=0) \]
Note:
Sometimes we use complex-valued solutions even if the initial and boundary solutions are real, for convenience.

If \( w = u + iv \) solves \( \Delta w = 0 \),
\( u \) and \( v \) are (real-valued) solutions as well.

E.g., \( u_{tt} = c^2 u_{xx} \)
has solutions of the form of plane waves
\[
U(x,t) = A e^{i(kx - \omega t)}
\]
wave number
wave frequency
\(
\omega = \frac{2\pi}{\lambda}
\)
wave length
frequency = \( \frac{2\pi}{\text{period}} \)

Homework: What is the relation between \( \omega, k \) and \( c \)
(called dispersion relation)
In real life solutions look like
\[
\sin(kx) \cos(\omega t)
\]
but there are four combinations!
Change of Variables

\[ \dot{U}_t = U_x^2 + U_{xx} - \text{nonlinear} \]

Define new dependent variable

\[ W = \exp(U) \]

**First way**

\[ a) \quad \dot{W}_t = e^U \cdot \dot{U}_t = W \dot{U}_t = \]

\[ = W (U_x^2 + U_{xx}) \]

\[ b) \quad W_x = W U_x \]

\[ c) \quad W_{xx} = W_x U_x + W U_{xx} \]

\[ = W U_x^2 + W U_{xx} \]

\[ = W (U_x^2 + U_{xx}) = \dot{W}_t \]

So in the new variable

\[ \dot{W}_t = W_{xx} - \text{heat equation} \]

becomes a linear PDE

\[ \dot{U} = \ln(W) \]

[But is \( W > 0 \) ?]

gives us solution in original notation
Second way

\[ u = \ln(w), \quad w > 0 \]

(a) \[ u_t = \frac{1}{w} w_t \]

(b) \[ u_x = \frac{1}{w} w_x \]

(c) \[ u_{xx} = \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2 \]

\[ u_t = u_x^2 + u_{xx} \implies \]

\[ \frac{1}{w} w_t = \frac{1}{w^2} w_x^2 + \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2 \]

\[ \implies \frac{1}{w} w_t = \frac{1}{w} \cdot w_{xx} \implies \]

\[ w_t = w_{xx} \text{ since } w > 0 \]

(challenge) Harder example: Cole-Hopf transform

(Heat) linear \[ \psi_t = \psi_{xx} \]

if \[ u = -2 \frac{\partial}{\partial x} (\ln \psi) \]

(Burgers) non-linear \[ u_t + uu_x = u_{xx} \]