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PDE SPRING 2018, A. DONEV

LECTURE 3

PDE Classification:

Linear PDEs

Let's introduce the concept of an

Operator: A "function of a function" or a function with an infinitely many arguments, or an infinite-dimensional generalization of a function.

$$\mathcal{L}(u) \quad \text{or} \quad \mathcal{L}\underbrace{[u(\cdot)]}_{\text{functional notation}}$$

caligraphic L

For linear PDEs we use:

$$\mathcal{L}(u) \equiv \underbrace{\mathcal{L}u}_{\text{Looks like matrix-vector multiplication}}.$$

(2) Examples

$$\begin{cases} \mathcal{L}u = ku_{xx} & \\ \mathcal{L}u = \nabla^2 u & \leftarrow \text{Laplacian} \\ \mathcal{L}u = \nabla\cdot u & \leftarrow \text{Divergence} \\ \mathcal{L}u = \nabla u & \leftarrow \text{Gradient} \\ \mathcal{L}u = u_t - ku_{xx} & \end{cases}$$

This is just <u>shorthand notation</u>

If $\mathcal{L}$ is the Laplacian operator, the heat equation can be written as

$$u_t = \mathcal{L}u$$

Other shorthand notation:

$$\begin{cases} \mathcal{L} = \partial_{xx} & \Rightarrow \mathcal{L}u = u_{xx} \\ \mathcal{L} = \nabla^2 & \Rightarrow \mathcal{L}u = \nabla^2 u \\ \mathcal{L} = \partial_t - k\,\partial_{xx} & \Rightarrow \mathcal{L}u = u_t - ku_{xx} \\ \mathcal{L} = \partial_{tt} - c^2\partial_{xx} & \underline{\text{etc}}. \end{cases}$$

Can be hard to read or <u>ambiguous</u> but it's convenient

(3) $\mathcal{L}$ is a <u>linear operator</u> if

SCALING $\rightarrow$ $\mathcal{L}(\alpha u) = \alpha\, \mathcal{L}(u)$, $\alpha \in \mathbb{R}, \mathbb{C}$

Linearity

SUPERPOSITION $\rightarrow$ $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

E.g. $\mathcal{L} = \partial_{xx}$ $\rightarrow$ follows from basic calculus

But $\mathcal{L}(u) = u u_x$ is <u>not</u> linear:

$$\mathcal{L}(\alpha u) = (\alpha u)(\alpha u)_x = \alpha^2 u u_x$$

$$\Rightarrow \mathcal{L}(\alpha u) = \alpha^{2} \mathcal{L}(u)$$

so this is of quadratic order and not linear.

It means Burger's equation is not linear.

Standard <u>Dirichlet</u> and <u>Neumann</u> BCs are also linear. E.g.

$$u(x=0,t) = f \Rightarrow (\alpha u)(x=0,t) = \alpha f$$

$$u_x(x=0,t) = f \Rightarrow (\alpha u)_x(x=0,t) = \alpha u_x = \alpha f$$

and similarly for addition

Linear PDEs

(4)

Definition: A PDE is *linear* if it has the form

$$\mathcal{L}u = f(x,t)$$

with $\mathcal{L}$ a linear operator, and the BCs are linear.

If $f = 0 = \mathcal{L}u$ we call the PDE *homogeneous*, otherwise inhomogeneous. To be homogeneous also *all* boundary and initial data must be zero.

E.g.

$$0 \le x, y \le 1$$

unit square

$$\begin{cases} \mathcal{L}u = 0 = u_{xx} + u_{yy} \\ u(x=0, y) = 0 \\ u(x=1, y) = 0 \\ u_x(x, y=0) = 0 \\ u_x(x, y=1) = 0 \end{cases}$$

$\leftarrow$ *ALL* must be zero

Abstract notations:

$$\begin{cases} \mathcal{L}u = f \\ \mathcal{B}u = g \end{cases}$$

$\uparrow$ linear operator

or just $\mathcal{L}u = f$ for short

$\uparrow$ *BOTH* PDE equation and BCs & ICs

(5)

$\mathcal{L}$ = eq + BCs + ICs

the most important property of linear PDEs is the

Superposition Principle

(1) If u_1 and u_2 are two solutions of a linear PDE
$\mathcal{L}u = 0$ (homogeneous)
then so is any linear combination

$$u = \alpha u_1 + \beta u_2 \quad , \quad \alpha, \beta \in \mathbb{C}$$

Since

$$\mathcal{L}u = \mathcal{L}(\alpha u_1) + \mathcal{L}(\beta u_2) =$$
$$= \alpha(\underbrace{\mathcal{L}u_1}_{=0}) + \beta(\underbrace{\mathcal{L}u_2}_{=0}) = 0$$

(2) If $\tilde{u}$ is a "particular" solution of the inhomogenous

$$\mathcal{L}u = f$$

then if $\mathcal{L}v = 0$,

$u = \tilde{u} + \alpha v$ is also a solution of $\mathcal{L}u =$ $\mathcal{L}(\tilde{u} + \alpha v) =$

$$= \mathcal{L}\tilde{u} + \alpha \mathcal{L}v = \mathcal{L}\tilde{u} = f$$

Q: $\boxed{?\ Ax = 0\ ?}$

(6) A consequence of #1 + #2 is:

(3) $\mathcal{L}u = f$ has a unique solution
<u>iff</u> $v = 0$ is the only
solution of $\mathcal{L}v = 0$

trivial solution

<u>Proof</u>:

Suppose there are two solutions
u_1 and u_2

$$\begin{cases} \mathcal{L}u_1 = f \\ \mathcal{L}u_2 = f \end{cases} \Rightarrow \mathcal{L}(u_1 - u_2) = 0$$

(a) If $\mathcal{L}u = 0$ has only $u = 0$ as the solution, then $u_1 = u_2$
so there is only one solution

(b) If u_1 and u_2 were two distinct solutions (so not unique)
then $u_1 - u_2 \neq 0$ and $\mathcal{L}u = 0$ has a nontrivial solution

<u>Conclusion</u>:

Linear BVPs have zero, one, <u>or</u> infinitely many solutions

Compare to $Ax = 0$ in
linear algebra class

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Well-posedness

A BVP that has a unique solution which varies continuously with the initial and boundary data is well-posed. All other BVPs are ill-posed

$$\boxed{\begin{gathered}\text{well-posed} = \\ \text{existence} + \text{uniqueness} + \text{stability}\end{gathered}}$$

Here is what stability means. Take

$$\mathcal{L} u = f$$

and perturb the "data" f:

$$\mathcal{L}(u + \underbrace{\delta u}_{\text{perturbation of the solution}}) = f + \delta f$$

Stability implies that

$$\boxed{\text{If } \delta f \text{ is small, so is } \delta u}$$

Small perturbations of the data lead to small perturbations of the solution.

$$\boxed{\|\delta u\| \leq C \|\delta f\|}$$

Here $\|f\|$ denotes the norm of the function f.

An example is the L_2 norm

$$\|f\|_2 = \sqrt{\int_{\text{Domain}} |f|^2 \, dx}$$

which is like the Euclidean vector norm

$$\|X\|_2 = \sqrt{X_1^2 + X_2^2 + \ldots + X_n^2}$$

There are other norms and the choice of norm matters — we will come back to this.

Example: Heat equation

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}$$

is well-posed (we will prove this later on)

But "backward" heat eq.

$$\begin{cases} u_t = -u_{xx} \\ u(x,0) = f(x) \end{cases}$$

is NOT well-posed

<u>Concrete notation</u>
(less confusion)

$x \in \mathbb{R}$
$t \in \mathbb{R}^+$

$$\begin{cases} u_t = -u_{xx} \\ u(x,0) = f(x) \end{cases}$$

<u>Abstract</u>
<u>notation</u>
(more elegant & general)

$$\begin{cases} \mathcal{L}u = g \\ \mathcal{L}u = \begin{Bmatrix} u_t + u_{xx} \\ u(x,0) \end{Bmatrix} \\ g = \begin{Bmatrix} 0 \\ f(x) \end{Bmatrix} \end{cases}$$

not the same "f" so I used g for one

$$\begin{cases} (u+\delta u)_t = -(u+\delta u)_{xx} \\ (u+\delta u)(x,0) = f(x)+\delta f(x) \end{cases}$$

Subtract first from second to get

$$\begin{cases} (\delta u)_t = -(\delta u)_{xx} \\ \delta u(x,0) = \delta f(x) \end{cases}$$

$$\begin{cases} \mathcal{L}u = g \\ \mathcal{L}(u+\delta u) = g+\delta g \\ \Rightarrow \mathcal{L}\delta u = \delta g \end{cases}$$

One example is

$$\delta u = u_n = \alpha \cos(nx) e^{n^2 t}$$

(Homework 1 shows this is a solution)

$$\Rightarrow \delta f = f_n = u_n(x, t=0) = \alpha \cos(nx)$$

$$\|\delta u\| = e^{n^2 t} \|\alpha \cos(nx)\| = e^{n^2 t} \|\delta f\|$$

(9) If $\|\delta u\| = e^{n^2 t}\|\delta f\|$

$$\|\delta u\|(t) = C_n(t)\,\|\delta f\|$$

$$C_n(t) = e^{n^2 t}$$

but $C_n \to \infty$ for <u>any</u> $t > 0$ if $n \to \infty$.

So we can make the solution differ by an <u>arbitrarily</u> large amount at any <u>positive</u> time

The PDE has no stability and is ill-posed. the solution exists and <u>is</u> unique but <u>not</u> stable.

hand waving intuition: <u>Physics</u>: The direction of time is set and time cannot be reversed (second law of thermodynamics)

One can add an inhomogeneous term:

$$\begin{cases} u_t = u_{xx} + g(x) \\ u(x,0) = f(x) \end{cases}$$

↑ this <u>is</u> well-posed

<u>Abstract notation</u>:

$$\mathcal{L}u = h$$

$$\mathcal{L}u = \begin{Bmatrix} u_t - u_{xx} \\ u(x,0) \end{Bmatrix}$$

$$h = \{g(x),\ f(x)\}$$

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Continuum superposition $\boxed{\mathcal{L}u=0}$

Let $u(x,t;\eta)$ be a solution, where η = parameter.

E.g. $u(x,t;n) = e^{-n^2 t}\cos(nx)$ for the heat equation

Then: coefficient or weight

$$v = \int_{\eta \in I} c(\eta)\, u(x,t;\eta)\, d\eta$$

greek "eta"

is also a solution for sufficiently smooth $c(\eta)$

Proof:

$$\mathcal{L}v = \mathcal{L}\left[\int_{\eta} c(\eta)\, u(x,t;\eta)\, d\eta\right]$$

can move inside if integrand smooth

$$= \int_{\eta} c(\eta)(\mathcal{L}u)\, d\eta = \int_{\eta} c(\eta)\, 0\, d\eta = 0$$

complex numbers ~~[Aside: If $w = u + iv$ is a solution, u and v are real solutions~~

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Example from Lecture 2

$$u(x,t;s) = \frac{1}{\sqrt{t}} e^{-\frac{(x-s)^2}{4t}}$$

shift or origin

is a solution of heat equation on $(\mathbb{R}, \mathbb{R}^+)$, for any s.

$\Rightarrow$ So is the continuous superposition:

(*) $$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4kt}} ds$$

Algebra is now easy!

We will show later that by a suitable choice of f we can solve the IVP for <u>any</u> initial condition. In fact (sneak preview / spoiler)

$$f(x) = u(x, t=0)$$

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Note:

Sometimes we use complex-valued solutions even if the initial and boundary solutions are real for convenience.

If $w = u + iv$ solves $\mathcal{L}w = 0$, u and v are (real-valued) solutions as well

E.g. $u_{tt} = c^2 u_{xx}$

has solutions of the form of plane waves

$$u(x,t) = A e^{i(kx - \omega t)}$$

wavenumber (k)

wavefrequency (ω)

$$k = \frac{2\pi}{\text{wavelength}} \qquad \omega = 2\pi \cdot \text{frequency} = \frac{2\pi}{\text{period}}$$

Homework: What is the relation between ω, k and c (called dispersion relation)

In real life solutions look like

$$\sin(kx)\cos(\omega t)$$

but there are four combinations!

Change of Variables

$$u_t = u_x^2 + u_{xx}$$ – nonlinear

Define new dependent variable

$$w = \exp(u)$$

First way

a $$w_t = e^u \cdot u_t = w u_t =$$
$$= w\left(u_x^2 + u_{xx}\right)$$

b $$w_x = w u_x$$

c $$w_{xx} = w_x u_x + w u_{xx}$$
$$= w u_x^2 + w u_{xx}$$
$$= w\left(u_x^2 + u_{xx}\right) = w_t$$

So in the new variable

$$w_t = w_{xx}$$ – heat equation

becomes a linear PDE

$$u = \ln(w)$$ $\boxed{\text{But is } w > 0\text{ ?}}$

gives us solution in original notation

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<u>Second way</u>

$u = \ln(w) \quad , w > 0$

a $u_t = \frac{1}{w} w_t$

b $u_x = \frac{1}{w} w_x$

c $u_{xx} = \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2$

$u_t = u_x^2 + u_{xx} \quad \Rightarrow$

$$\frac{1}{w} w_t = \frac{1}{w^2} w_x^2 + \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2$$

cancel

$$\Rightarrow \frac{1}{w} w_t = \frac{1}{w} \cdot w_{xx} \Rightarrow$$

$$w_t = w_{xx} \quad \text{since} \quad w > 0$$

(challenge) <u>Harder</u> example : <u>Cole-Hopf transform</u>

(Heat) linear $\begin{cases} \varphi_t = \varphi_{xx} \\ \quad ||| \\ u_t + u u_x = u_{xx} \end{cases}$ if $u = -2 \frac{\partial}{\partial x} (\ln \varphi)$

(Burgers) non-linear