In this lecture we will fill in a few missing pieces for the heat equation.

**Inhomogeneous BCs**

\[
\begin{cases}
    u_t = \nabla^2 u & \text{in } \Omega \\
    u(\partial \Omega) = \psi(\partial \Omega) \neq 0 & \text{inhomogeneous Dirichlet} \\
    u(\vec{x}, t=0) = u_0(\vec{x}) & 
\end{cases}
\]

Assume we could solve the Laplace equation with the same BCs for \( \psi(\vec{x}) \):
\[
\nabla^2 \psi = 0, \quad \psi(\partial \Omega) = \psi(\partial \Omega)
\]
New, writing

\[ u = \varphi + w(x,t) \]

\[ \Delta^2 u = \Delta^2 \varphi + \Delta^2 w = \Delta^2 w = 0 \]

\[ u_t = w_t \Rightarrow w_t = \Delta^2 w \]

BCs
\[ u(\partial \Omega) = \varphi(\partial \Omega) + w(\partial \Omega) = 0 \]

ICs
\[ w(\partial \Omega) = 0 \]
\[ w(0) = 0 \]

we get

\[ w_t = \Delta^2 w \]

the PDE

\[ \begin{cases} 
    w_t = \Delta^2 w \\
    w(\partial \Omega) = 0 - \text{Homogeneous!} \\
    w(0) = u_0 - \varphi 
\end{cases} \]

which we know how to solve by using the eigenvalues and eigenfunctions of the Laplacian on \( \mathbb{R}^2 \).
\[ \begin{align*}
\sum A_n^2 u_n &= \lambda_n u_n \\
\sum u_n(\infty) &= 0
\end{align*} \]

(normalized) [orthonormal]

gives a complete basis

\[ u = \sum_{n=1}^{\infty} A_n(t) u_n(x) + \Phi(x) \]

\[ u_t = \frac{\partial^2}{\partial x^2} u \quad \text{becomes} \]

\[ \sum A_n' u_n = \sum \lambda_n A_n u_n \]

\[ A_n' = \lambda_n A_n \Rightarrow A_n = A_n(0)e^{\lambda_n t} \]

\[ u(x^2, t) = c\Phi(x) + \sum_{n=1}^{\infty} A_n(0)e^{\lambda_n t} u_n(x^2) \]

Initial condition gives

\[ u_0(x^2) = \sum_{n=1}^{\infty} A_n(0) u_n(x^2) + \Phi(x) \]
\[ \sum_{n} A_n(0) u_n(x^2) = u_0(x) - u(x) = u(x) \]

Because the \( u_n \)'s are orthonormal

\[ A_n(0) = (u_n, u_n) = \int_{-1}^{1} u_n(x) \overline{u_n(x)} dx \]

which completes the solution.

Note that this method would not quite work if the BCs were time-dependent!

\[ u(x, t) = \varphi(x, t) \]

because now \( \varphi = \varphi(x, t) \)

and so \( u = \varphi + w \)

\[ u_t = w_t + \varphi_t \neq w_t \]

\[ \Rightarrow \left\{ \begin{array}{l}
  w_t = \nabla^2 w - i n t \quad \text{source term} \\
  \varphi(\Omega) = 0 \\
  \varphi(x, 0) = u_0(x) - u(x, 0)
\end{array} \right. \]

which is diffusion with a source term \( \to \) harder
Sources

Let us now consider the heat equation with sources

\[ \begin{align*}
    u_t &= ku_{xx} + f(x, t) \\
    u(0, t) &= u(\pi, t) = 0, \quad t > 0 \\
    u(x, 0) &= 0, \quad 0 < x < \pi
\end{align*} \]

We will first solve this using eigenfunctions.

Expand both the solution and the forcing into an infinite series in the eigenfunctions of the Laplacian with homogeneous Dirichlet BCs:

\[ \begin{align*}
    u(x, t) &= \sum_{n=1}^{\infty} g_n(t) \sin(nx) \\
    f(x, t) &= \sum_{n=1}^{\infty} f_n(t) \sin(nx)
\end{align*} \]

Fourier sine series
We know
\[ f_n(t) = \frac{2}{\pi} \int_0^\pi f(x, t) \sin(nx) \, dx \]
so all we need is to find the coefficients \( g_n(t) \)

\[ u_t = \sum_n g_n \sin(nx) \]

\[ u_{xx} = \sum_n \lambda_n g_n \sin(nx) \]

where \( \lambda_n = n^2 \)

\[ \text{(PDE)} \quad u_t - u_{xx} = f \]

\[ \sum_{n=1}^{\infty} (g_n + kn^2g_n) \sin(nx) = \sum_{n=1}^{\infty} f_n(t) \sin(nx) \]

\[ \Rightarrow \text{ (by orthogonality & linear independence of \( \sin(nx) \))} \]

\[ g_n(t) + kn^2g_n(t) = f_n(t) \]

which is now an ODE, one ODE per eigenvector, and easy to solve.
\[ g_n(t) = g_n(0) e^{-n^2 k t} + \int_0^t f_n(\tau) e^{-n^2 k (t-\tau)} \, d\tau \]

Recall Duhamel's principle.

Initial condition gives

\[ u(x, t=0) = \sum g_n(0) \sin(nx) = 0 \]

\[ \Rightarrow g_n(0) = 0 \]

And finally we get the solution

\[
\left\{
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} \left( \int_0^t f_n(\tau) e^{-n^2 k (t-\tau)} \, d\tau \right) \sin(nx) \\
  f_n(\tau) &= \frac{2}{\pi} \int_0^\pi f(x, \tau) \sin(nx) \, dx
\end{align*}
\right.
\]

This is nothing other than Duhamel's principle for the PDE itself \( \text{Example 4.25} \) in APDE.