PDE Spring 2016

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Lecture 18

Heat Equation revisited

In this lecture we will fill in a few missing pieces for the heat equation.

Inhomogeneous BCs

$$\begin{cases} u_t = \nabla^2 u & \text{in } \Omega \\ u(\partial\Omega) = \varphi(\partial\Omega) \neq 0 \\ \quad \nwarrow \text{ inhomogeneous Dirichlet} \\ u(\vec{x}, t=0) = u_0(\vec{x}) \end{cases}$$

Assume we could solve the Laplace equation with the same BCs for $v(x)$

$$\nabla^2 v = 0 \quad , \quad v(\partial\Omega) = \varphi(\partial\Omega)$$

Now, writing

$$u(x,t) = \vartheta(x) + w(x,t)$$

$$\nabla^2 u = \underbrace{\nabla^2 \vartheta}_{\text{zero}} + \nabla^2 w = \nabla^2 w =$$

$$= u_t = w_t \Rightarrow w_t = \nabla^2 w$$

BCs

$$u(\partial\Omega) = \vartheta(\partial\Omega) + w(\partial\Omega) =$$

$$= \varphi(\partial\Omega) + w(\partial\Omega) = \varphi(\partial\Omega)$$

ICs $\quad w(\partial\Omega) = \varphi(\partial\Omega)$

$$u(\vec{x},0) = \vartheta(\vec{x}) + w(\vec{x},0) = u_0(\vec{x})$$

$$w(x,0) = u_0(x) - \vartheta(x)$$

we get the PDE

$$\begin{cases} w_t = \nabla^2 w \\ w(\partial\Omega) = 0 - \text{Homogeneous!} \\ w(0) = u_0 - \vartheta \end{cases}$$

which we know how to solve by using the eigenvalues and eigenfunctions of the Laplacian on Ω

$$\begin{cases} \nabla^2 u_n = \lambda_n u_n \\ u_n(\partial\Omega) = 0 \end{cases}$$ ③

normalized

gives a complete orthonormal basis

$\{u_1, u_2, \ldots\ldots\}$

$$u = \sum_{n=1}^{\infty} A_n(t)\, u_n(\vec{x}) + v(\vec{x})$$

$u_t = \nabla^2 u$ becomes $\begin{cases} v_t = 0 \\ \nabla^2 v = 0 \end{cases}$

$$\sum_n A_n' u_n = \sum_n \lambda_n A_n u_n$$

$$A_n' = \lambda_n A_n \Rightarrow A_n = A_n(0)\, e^{\lambda_n t}$$

$$\boxed{u(\vec{x}, t) = v(\vec{x}) + \sum_{n=1}^{\infty} A_n(0)\, e^{\lambda_n t}\, u_n(\vec{x})}$$

Initial condition gives

$$u_0(\vec{x}) = \sum_n A_n(0)\, u_n(\vec{x}) + v(\vec{x})$$

$$\sum_n A_n(0)\, u_n(\vec{x}) = u_0(x) - \vartheta(x) = \overline{u}(x)$$

$(u_n, v_n) = 1$

Because the u_n's are orthonormal ④

$$A_n(0) = (u_n, \overline{u}_{\blacksquare}) = \int_\Omega u_n(\vec{x})\, \overline{u}_0^*(\vec{x})\, d\vec{x}$$

which completes the solution.

Note that this method would not quite work if the BCs were time-dependent!

$$u(\vec{x} \in \partial\Omega) = \varphi(\vec{x}, t)$$

because now $\vec{\vartheta} \equiv \vec{\vartheta}(\vec{x}; t)$ ← parameter

and so $u = w + \vartheta$

$$u_t = w_t + \vartheta_t \neq w_t$$

$$\Rightarrow \begin{cases} w_t = \nabla^2 w - \vartheta_t \\ w(\partial\Omega) = 0 \\ w(\vec{x}, 0) = u_0(\vec{x}) - \vartheta(\vec{x}, 0) \end{cases}$$

source term (comes after solving Laplace eq.)

which is diffusion with a source term → harder

Sources

Let us now consider the heat equation with sources

$$\begin{cases} u_t = k u_{xx} + f(x,t) \\ u(0,t) = u(\pi,t) = 0 \quad , \quad t>0 \\ u(x,0) = 0 \quad , \quad 0 < x < \pi \end{cases}$$

Dirichlet

We will first solve this using eigenfunctions.

Expand <u>both</u> the solution and the forcing into an infinite series in the eigenfunctions of the Laplacian with homog. Dirichlet BCs:

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin(nx) \\ f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(nx) \end{cases}$$

Fourier sine series

We know

$$f_n(t) = \frac{2}{\pi} \int_0^\pi f(x,t) \sin(nx)\, dx$$

so all we need is to find the coefficients $g_n(t)$

$$u_t = \sum_n g_n' \sin(nx)$$

$$u_{xx} = \sum_n \lambda_n g_n \sin(nx)$$

where $\lambda_n = n^2$

(PDE) $u_t - u_{xx} = f$

$$\Rightarrow \sum_{n=1}^\infty (g_n' + k n^2 g_n) \sin(nx) = \sum_{n=1}^\infty f_n(t) \sin(nx)$$

$\Rightarrow$ (by orthogonality & linear independence of $\sin(nx)$)

$$\boxed{g_n'(t) + k n^2 g_n(t) = f_n(t)}$$

missing $g_n(0)$

which is now an ODE, one ODE per eigenvector, and easy to solve.

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$$g_n(t) = g_n(0)\, e^{-n^2 k t} + \int_0^t f_n(\bar\tau)\, e^{-n^2 k (t-\bar\tau)}\, d\bar\tau$$

Recall Duhamel's principle

Initial condition gives

$$u(x, t=0) = \sum g_n(0) \sin(nx) = 0$$

$\varphi(x)$

no additional complexity in nonzero IC

$$\Rightarrow \quad g_n(0) = 0$$

And finally we get the solution

$$\begin{cases} u(x,t) = \displaystyle\sum_{n=1}^{\infty} \left(\int_0^t f_n(\bar\tau)\, e^{-n^2 k (t-\bar\tau)}\, d\bar\tau \right) \sin(nx) \\ f_n(\bar\tau) = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x,t) \sin(nx)\, dx \end{cases}$$

This is nothing other than Duhamel's principle for the PDE itself → Example 4.25 in APDE