# PDE Spring 2016

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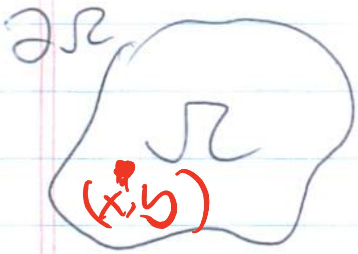
## Lecture 17

## The Poisson Equations & Laplace

Let's start from the Laplace equation

$u(x,y)$




$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u(\partial\Omega) = \varphi(\partial\Omega) \end{cases}$$

Dirichlet BCs

In one-dimension, the Laplace equation is trivial

$$u'' = 0 \Rightarrow$$

$$u = ax + b \quad \text{(linear)}$$

But in two and three dimensions it is much more interesting

Functions $u(x,y)$ that satisfy

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

are called <u>harmonic functions</u> and are central to <u>complex analysis</u>

The Laplace equation is often solved by <u>separation of variables</u>

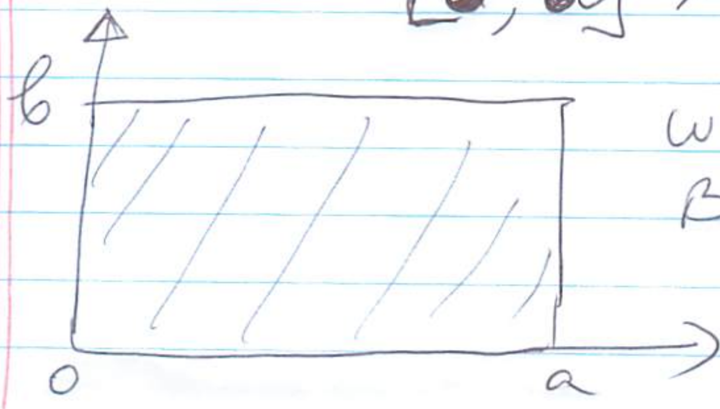
<u>E.g.</u>

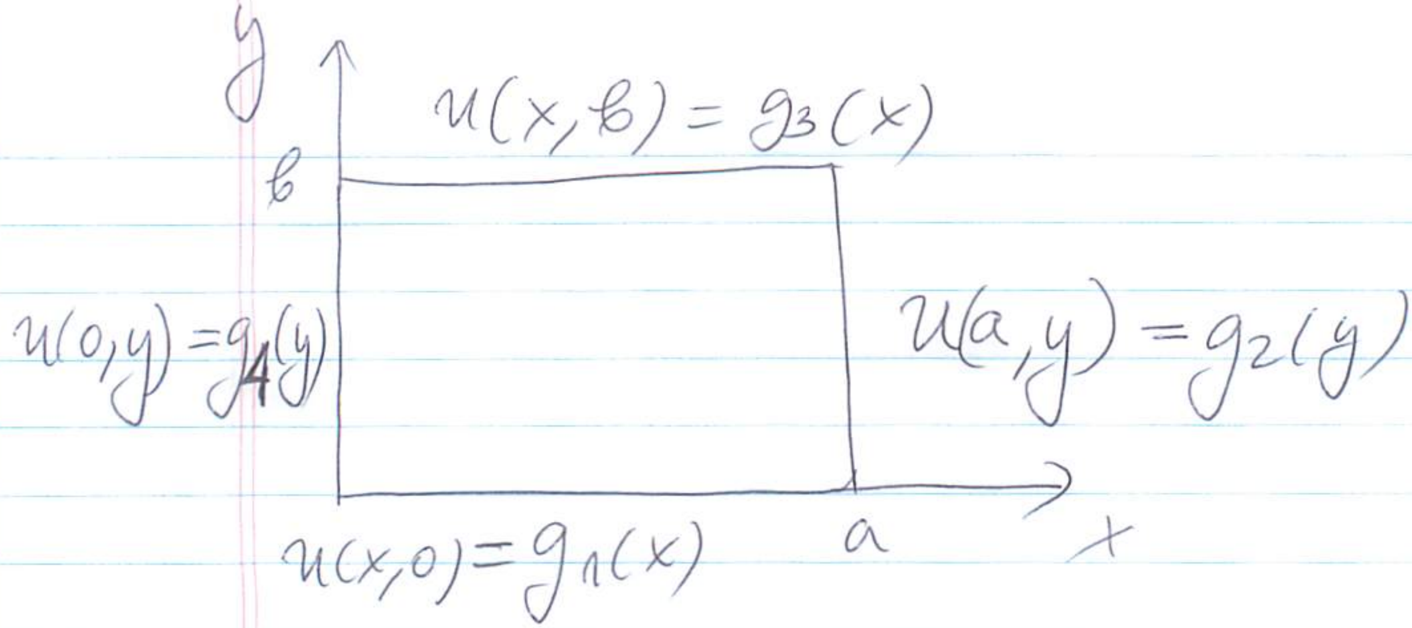
$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dl = 0 \text{ for Neumann}$$

Solve $\nabla^2 u = 0$ on rectangle

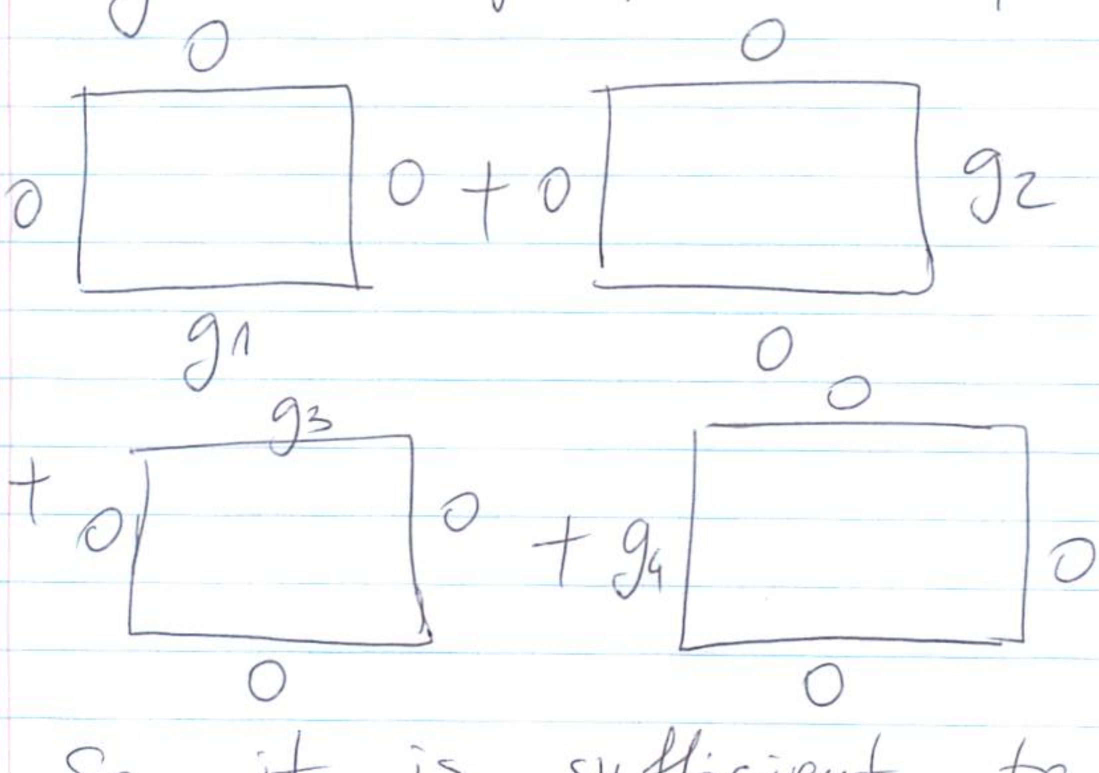
$$\Omega = [0,a] \times [0,b]$$

with inhomogeneous Dirichlet BCs on the sides of the rectangle






We can split this problem into four simpler subproblems by the superposition principle




So it is sufficient to consider non-homogeneous BCs on <u>only one side</u> of the rectangle

$$\begin{cases} \nabla^2 u = 0 \\ u(x,b) = u(a,y) = u(0,y) = 0 \\ u(x,0) = g(x) \end{cases}$$

$$\begin{cases} u(x,b) = 0 \\ X(x)\,\underline{Y(b)=0} \\ \qquad \forall x \end{cases}$$

Look for separable solutions

$$u(x,y) = X(x)\,Y(y)$$

$$\begin{cases} u(a,y) = 0 \\ X(a)\,Y(y) = 0 \\ \quad \forall y \end{cases}$$

$$\Rightarrow \begin{cases} X(0) = X(a) = 0 \\ Y(b) = 0 \end{cases}$$

but $Y(0)$ is undetermined

set $\boxed{Y(0) = 1}$

$$u_{xx} + u_{yy} = X''Y + XY'' = 0$$

$$\Rightarrow -\frac{X''}{X}(x) = \frac{Y''}{Y}(y) = \text{constant} = \underset{\underset{0}{\vee}}{\lambda}$$

We get the one-dimensional eigenvalue problem

$$\begin{cases} X'' = -\lambda X \\ X(0) = X(a) = 0 \end{cases}$$

which we have already solved

$$\begin{cases} X_n = \sin\left(\frac{n\pi x}{a}\right) \\ \lambda_n = \left(\frac{n\pi}{a}\right)^2 \end{cases}$$

$$Y'' = \lambda_n Y(y)$$

positive sign

$$Y = c_1 e^{\sqrt{\lambda_n} y} + c_2 e^{-\sqrt{\lambda_n} y}$$

$$Y(b) = c_1 e^{\sqrt{\lambda_n} b} + c_2 e^{-\sqrt{\lambda_n} b} = 0$$

$$\Rightarrow \quad c_1 = -c_2 e^{-2\sqrt{\lambda_n} b}$$

$$Y(y) = c_2 \left[ -e^{\sqrt{\lambda_n}(y-2b)} + e^{-\sqrt{\lambda_n} y} \right]$$

Usually written as

$$Y = \underbrace{C_2 e^{-\sqrt{\lambda_n} b}}_{C} \left[ -e^{\sqrt{\lambda_n}(y-b)} + e^{-\sqrt{\lambda_n}(y-b)} \right]$$

Denoting

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

we can write

$$Y = C \sinh\left(\sqrt{\lambda_n}\,(y-b)\right)$$

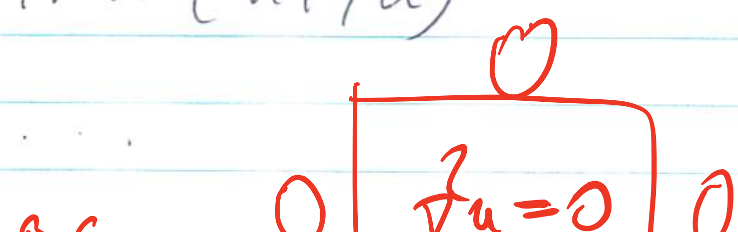
From

$$Y(0) = 1 = -C \sinh\left(\sqrt{\lambda_n}\, b\right)$$

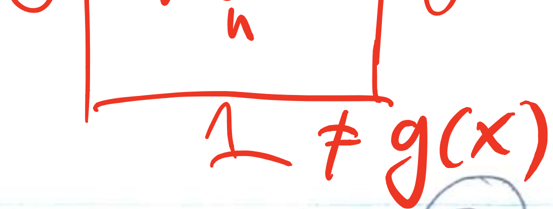
$$\Rightarrow C = -\frac{1}{\sinh\left(\sqrt{\lambda_n}\, b\right)}$$

Final answer for eigenfunctions

$$\left\{ \begin{array}{l} u_n(x,y) = \dfrac{\sinh\left(n\pi(1-y/b)/a\right)}{\sinh(n\pi/a)} \sin\left(\dfrac{n\pi x}{a}\right) \\ \\ n = 1, 2, \ldots \end{array} \right.$$



BCs:

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We hope we can expand the solution as a sum of these

$$u = \sum_{n=1}^{\infty} A_n u_n(x,y)$$

$$u(x,0) = g(x) \quad BC$$

$$\Rightarrow \quad g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

Fourier sine series

$$\Rightarrow$$

$$u = \sum_{n=1}^{\infty} \left[\frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx\right] \sin\left(\frac{n\pi x}{a}\right) \cdot \left[\frac{\sinh\left(n\pi(1-y/b)/a\right)}{\sinh\left(n\pi/a\right)}\right]$$

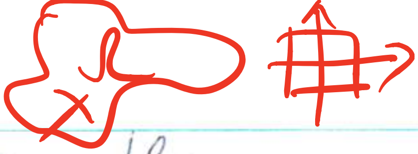
is the solution of Laplace's equation

Sidenote: $\Omega$ = circle

$$u(x,y) = u(r,\theta) = R(r)\,\Theta(\theta)$$

$$\nabla^2 u = \frac{1}{r}(ru')' + \ldots$$

What about potato





Now let's consider the Poisson equation on a rectangle

$$\begin{cases}\nabla^2 u = u_{xx}+u_{yy} = f(x,y)\\ u(\partial\Omega)=0 \quad \text{on boundary}\end{cases}$$

yes even for potato

$Ax=b$

Idea: To solve $\mathcal{L}u=f$ first find the eigenfunctions of $\mathcal{L}$, and then expand both solution and rhs in that basis

$$u=\sum_n a_n u_n$$

$$f=\sum_n b_n u_n$$

$$\Rightarrow \mathcal{L}u = \sum_n a_n\overbrace{\left(\mathcal{L}u_n\right)}^{\lambda_n u_n \,=} =$$

$$u_m \cdot \Big| \quad = \sum \lambda_n a_n u_n = f = \sum b_n u_n$$

$\Rightarrow$ by orthogonality

$$b_n = \lambda_n a_n \Rightarrow \boxed{a_n = \frac{b_n}{\lambda_n}}$$

This assumes that $\lambda = 0$ is not an eigenvalue, which is a sufficient and necessary condition for $\mathcal{L}u = f$ to have a unique solution (9)

$$\Rightarrow u = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} u_n$$

From orthogonality, from

$$f = \sum_n b_n u_n$$

$$\Rightarrow b_n = \frac{(u_n, f)}{(u_n, u_n)} = \int_\Omega u_n(x,y) f(x,y)\, dx\, dy \Rightarrow$$

$= (u_n, f)$ is orthonormal basis

$$\boxed{u = \sum_{n=1}^{\infty} \frac{(u_n, f)}{\lambda_n} u_n} \equiv \nabla^{-2} f$$

$Ax = \lambda x$ harder than $\underline{Ax = b}$




The finite-dimensional linear algebra version of this is:

$$Ax = b \Rightarrow x = A^{-1} b$$

$u^{-1} = u^*$

$$A = \mathcal{U} \Lambda \mathcal{U}^* \quad \text{if } A \text{ unitarily diagonalizable}$$

unitary

$u^{-1}$ $(u^*)^{-1}$

$$\Rightarrow A^{-1} = \mathcal{U} \Lambda^{-1} \mathcal{U}^*$$

Since $A A^{-1} = \mathcal{U} \Lambda \mathcal{U}^* \mathcal{U} \Lambda^{-1} \mathcal{U}^*$

$$= \mathcal{U}\mathcal{U}^* = I$$

$(AB)^{-1} = B^{-1}A^{-1}$

$$\Rightarrow x = A^{-1} b = \mathcal{U} \Lambda^{-1} \left(\underline{\mathcal{U}^* b}\right)$$

dot product with row of $U^*$ = column of $U$

We don't usually solve linear systems in $\mathbb{R}^n$ this way since there are faster ways, but for PDEs it is the best way to construct an "inverse Laplacian"

$$u = \nabla^{-2} f \qquad \text{(notation)}$$

$\nabla^2 u \,\hat{=}\, f$

The only missing piece is to construct the eigenfunctions of the Laplacian on ~~a~~ rectangle with homogeneous Dirichlet BCs. (11)

$$\nabla^2 u = -\cancel{M}u \quad , \quad u(\partial\Omega)=0$$

$mu$

Take

$u = \underline{\overline{X}}(x)\underline{\overline{Y}}(y)$ separable

(only works because domain is so simple)

$$\Rightarrow \quad -\frac{X''(x)}{X} = \frac{Y''(y)}{Y} - \mu = \lambda$$

$$\Rightarrow \begin{cases} X'' = -\lambda X \\ X(0)=X(a)=0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{a}\right)^2$$

$$X = \sin\left(\frac{n\pi x}{a}\right)$$

Now

$$\begin{cases} Y'' = (\underline{\mu-\lambda})Y \\ Y(0) = Y(b) = 0 \end{cases}$$

shifted eigenvalue

$$Y = \sin\left(\frac{m\pi y}{b}\right) \Rightarrow \mu - \lambda = \left(\frac{m\pi}{b}\right)^2$$

$$\Rightarrow \boxed{\mu_{m,n} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 > 0} \quad (12)$$

$n \geq 0 \quad m \geq 0$ <u>not</u> $n = m = 0$

$$u_{m,n} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

two integer indices

$$f = \sum_{m,n} b_{m,n}\, u_{m,n}$$

$$b_{m,n} = \frac{(f, u_{m,n})}{(u_{m,n}, u_{m,n})} =$$

$$b_{m,n} = \frac{\int_0^b\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy}{\left(\frac{1}{4}ab\right)}$$

and $$u = \sum_{m,n} \frac{b_{m,n}}{\mu_{m,n}}\, u_{m,n}$$

In this case the x and y direction completely separate

Note that the Poisson equation with inhomogeneous BCs is, by superposition, a sum of Laplace with inhomogeneous BCs and Poisson with homog. conditions (13)

$$\begin{cases} \nabla^2 u = \mathcal{L} u = f \\ u(\partial\Omega) = \varphi(\partial\Omega) \end{cases} \Rightarrow$$

$$u = u_1 + u_2$$

$$\begin{cases} \nabla^2 u_1 = f \\ u(\partial\Omega) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \nabla^2 u_2 = 0 \\ u_2(\partial\Omega) = \varphi \end{cases}$$

We will not prove it here but it is not hard to show that the Poisson & Laplace equations with Dirichlet BCs are well-posed, notably, the solution is unique (we already proved the maximum principle for Laplace's equation & this can be used in a proof)

# A sidenote: Green's Functions

Assume that we could find the Green's function for the Poisson equation with homogeneous BCs

$$\begin{cases} \nabla^2 G = \delta(x - x_0, y - y_0) \leftarrow \text{Point source} \\ (x_0, y_0) \in \Omega \\ G(\partial\Omega) = 0 \quad (\text{homogeneous Dirichlet}) \end{cases}$$

Since

$$f(x,y) = \iint_{(x_0,y_0)\in\Omega} f(x_0,y_0)\,\delta(x-x_0, y-y_0)\,dx_0\,dy_0$$

by superposition $\mathcal{L}u = f \Rightarrow$

$$u(x,y) = \iint_{(x_0,y_0)\in\Omega} f(x_0,y_0)\,G(x,y;x_0,y_0)\,dx_0\,dy_0$$

We usully write this in vector notation

$$G(x,y\,;\,x_0,y_0) \equiv G(\underset{\uparrow \atop \text{target}}{\vec{x}}, \underset{\uparrow \atop \text{source}}{\vec{x}_0})$$

$$u(\vec{x}) = \int_{\Omega} G(\vec{x},\vec{y})\, f(\vec{y})\, d\vec{y}$$

Recall our previous formula

$$u(\vec{x}) = \sum_{n=1}^{\infty} \frac{(f, u_n)}{\lambda_n (u_n, u_n)}\, u_n(\vec{x})$$

$$= \sum_{n=1}^{\infty} \frac{1}{(\lambda_n \|u_n\|_2^2)} \left[\int_{\Omega} f(\vec{y})\, u_n(\vec{y})\, d\vec{y}\right] u_n(\vec{x})$$

$$= \int_{\Omega} \underbrace{\left(\sum_{n=1}^{\infty} \frac{u_n(\vec{y})\, u_n(\vec{x})}{\lambda_n \|u_n\|_2^2}\right)} f(\vec{y})\, d\vec{y}$$

<u>Defines</u> $G(\vec{x},\vec{y})$