Sturm-Liouville Problems

As motivation let us consider the heat equation on a circle with a radially-symmetric IC & BC

\[ u_{tt} = u_{xx} + u_{yy} = \Delta u \]

for \( x^2 + y^2 < R^2 \)

\[ u(\partial R) = 0 \] \[ \Phi(R) \]

\[ u(R, t=0) = \Phi(R) \]

We know that the solution is spherically-symmetric so we should use polar coordinates
\[
\begin{align*}
\begin{cases}
  x &= r \cos \theta \\
  y &= r \sin \theta 
\end{cases} \\
-\Delta u &= - \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \\
&= \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right)
\end{align*}
\]
(from Calculus 3)

For us, \( u = u(r) \) only, so the eigenvalue problem we need to solve is

\[ \Delta u = \lambda u \]

where

\[ (\Delta u)(r) = - \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \]

\[ = - \frac{1}{r} \left( ru' \right)' = \lambda u \]

So we get

\[ u'(r) = \frac{\partial u}{\partial r}(r) \]

\[ \frac{1}{r} \left( ru' \right)' + \lambda u = 0 \]

\[ u(0) = u(1) = 0 \]

which is a one-dimensional problem (same in 3D but different coefficients)
These types of problems appear often in practice and we consider them in this class.

We consider the two-point BVP (i.e. ODE!)

$$\lambda u = \begin{cases} 
\Delta u = -(p(x)u'(x))' + q(x)u \\
= -pu'' - p'u' + qu = f(x)
\end{cases}$$
on $a \leq x \leq b$, with either Dirichlet, Neumann, periodic, or mixed BCs.

There are two cases to be considered:

1. Non-singular:
   \[ p(x) > 0 \quad q(x) > 0 \quad \forall a < x < b \]
   \[ \text{cont. differentiable} \quad \text{continuous} \]

2. Singular (harder!)
   \[ p(a) = 0 \quad \text{or} \quad p(b) = 0 \]
Observe that a general BVP in one dimension that is linear and second-order takes the form:

\[ u'' = a(x)u' + b(x)u - g(x) \]

With a trick, we can write this in the SL form:

\[ p(x)u'' - p(x)a(x)u' - p(x)b(x)u = p(x)g(x) \]

From \((pu')' = pu'' + p'u'\) we get \(pu'' = (pu')' - p'u'\)

\[ \Rightarrow (pu')' + (pa + p')u' + pbu = pg \]

versus the SL equation \((pu')' + qu = f(x)\)

So we want \(pa + p' = 0\)
\[ p'(x) = -axp(x) \Rightarrow \]
\[ p(x) = \exp \left[ -\int a(x) \, dx \right] > 0 \]
\[ q(x) = p(x)g(x) > 0 \]
\[ f(x) = -p(x)g(x) \]

Observe that as long as \( a(x) \) is integrable, \( p(x) \) exists and is positive, so the Sturm-Liouville problem is quite general!

The key observation that will allow us to solve SL problems is to find the SL eigenfunctions \& eigenvalues

\[ \mathcal{L}u = \lambda u + \text{BCs} \]

and then expand the RHS \( f(x) \) or initial condition for IVPs into an orthogonal series
The key observation is that:

The SL operator is an self-adjoint operator complex conjugate

\[
(u, \mathcal{L}v) = \int_a^b u \left[ -\left( pv \bar{v} \right)' + q v \bar{v} \right] dx
\]

\[
= \int_a^b \left( u \left( pv \bar{v}' \right)' \right) dx + \int_a^b q u \bar{v} dx
\]

\[
= -\left[ pu \bar{v} \right]_a^b - (pu')' \bar{v} dx
\]

\[
+ \int_a^b \left( pu \bar{v} + qu \bar{v} \right) dx
\]

\[
= \left[ p \left( n' \bar{v} - u \bar{w} \right) \right]_a^b + ( Xu, \psi)
\]
Therefore
\[(u, Xv) - (Xu, v) = \left[ p(u^{\dagger}v - v^{\dagger}u) \right]_a\]

this vanishes for many BCs

So for a number of common BCs we have
\[(u, Xv) = (Xu, v) \Rightarrow X^* = X \text{ self-adjoint}\]

From the fact \(X\) is a symmetric (self-adjoint) operator we already knew a lot of consequences.

E.g. it's eigenvalues are real, eigenvectors are orthogonal (even complete), etc.

Turns out we know their sign also.
Theorem:

If \( p(x) > 0 \), \( q(x) > 0 \) for \( x \in (a, b) \), then the eigenvalues are real and positive. This means the operator is symmetric positive definite.

Since \( \lambda = 0 \) is not an eigenvalue, we know

\[ \lambda u = 0 \]

has only \( u = 0 \) as the solution. This means

\[ \lambda u = f(x) \]

has a unique solution why? i.e. the SL two-point BVP is well-posed.
To prove $\lambda$ are real we only need $\lambda^* = \lambda$ and we proved this already.

To prove they are positive take

$$(u, \lambda u) = \lambda (u, u)$$

integrate by parts

$$\int_a^b \left( -u (pu')' + q u u' \right) \, dx$$

$$= \int_a^b \left( p |u'|^2 + q (u)^2 \right) \, dx > 0$$

+ vanishing boundary terms

$$\Rightarrow \lambda = \frac{\int_a^b \left( p |u'|^2 + q |u|^2 \right) \, dx}{\int |u|^2 \, dx}$$

Rayleigh quotient

$$\Rightarrow \lambda > 0$$
We in fact know a few more things (not proven here):

1. The eigenvalues are simple (not repeated) and there are countably many of them with larger $k$ decays faster:
   
   $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$

   and $\lambda_k \to +\infty$ for $k \to \infty$

2. All eigenvectors are orthogonal (we have proven this)

3. The eigenvectors form a complete $L^2$ basis, i.e.,

\[ \forall f(x) \in L^2 \]

\[ f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \]

\[ n=1 \]

\[ (u_n, f) \]

converges in norm