PDE Spring 2016

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Lecture 15

Convergence of Fourier Series

Over the past few lectures we set up a rather general separation of variables procedure for solving BVPs of the form:

$$u_t = \underset{+BC}{\mathcal{L} u} \quad , \quad u(x,0) = \varphi(x)$$

Namely, we first solve the eigenvalue problem

$$\mathcal{L} u = \lambda u \quad + \text{boundary conditions}$$

and find all of the eigenvectors (eigenfunctions) and eigenvalues

$$\begin{cases} \lambda_1, \lambda_2, \ldots & \left(\text{e.g. } \left(\frac{n\pi}{l}\right)^2\right) \\ u_1, u_2, \ldots & \left(\text{e.g. } \sin\left(\frac{n\pi x}{L}\right)\right) \end{cases}$$

If we can expand the initial condition in the eigenbasis of $\mathcal{L}$,

$$\varphi(x) = u(x,0) = \sum_{k=0}^{\infty} A_k\, u_k(x) \quad \Big| \cdot u_m(x)$$

δ_{km}

then we know how to find the A_k's since we know u's belonging to distinct eigenvalues are orthogonal and those belonging to the same eigenvalue can be made orthogonal

$$A_k = \frac{(u_k, \varphi)}{(u_k, u_k)}$$

← integral = formula involving k

Furthermore, if we start with initial condition

$$\varphi \equiv u_k$$

$A_k = 1$ all others zero

then we know it decays exponentially

$$u(x,t) = a_k(t)\, u_k(x)$$

$a_k(t=0) = A$

$$u_t = \underline{a_k'(t)}\, \cancel{u_k} = \mathcal{L}[a_k u_k]$$

$$= \underline{a_k(t)\, \lambda_k}\, \cancel{u_k} \quad \Longrightarrow \mathcal{L} u_k = \lambda_k u_k$$

$$a_k'(t) = \lambda_k a_k \Rightarrow$$

$$a_k(t) = e^{\lambda_k t}\, \overbrace{a_k(0)}^{A_k}$$

Therefore, by the superposition principle the solution of

$$\begin{cases} u_t = \mathcal{L} u \\ u(t=0) = \varphi(x) \end{cases}$$

is

$$u(x,t) = \sum_k \overset{A_k}{\overset{\shortparallel}{a_k(0)}}\, e^{\lambda_k t}\, u_k(x)$$

$$\boxed{u(x,t) = \sum_k \frac{(u_k, \varphi)}{(u_k, u_k)}\, e^{\lambda_k t}\, u_k(x)}$$

This gives us the solution as an infinite series but it all rested on an assumption that φ could be expanded as an (infinite) sum of eigenfunctions.

$\boxed{\text{For what } \varphi(x) \text{ is this possible?}}$

If this were a finite-dimensional system of ODEs

$$\frac{d\vec{x}(t)}{dt} = \overleftrightarrow{A}\,\vec{x}(t) \quad \longleftarrow \; u_t = \mathcal{L}u$$

and A were symmetric/Hermitian, it would be unitarily diagonalizable and the same procedure would work

$$\vec{x}(t) = \sum_{k=1}^{n} \frac{(\vec{x}_k \cdot \vec{x}(0))}{(\vec{x}_k \cdot \vec{x}_k)} e^{\lambda_k t} \vec{x}_k(0)$$

where x_k are the eigenvectors and λ_k are the eigenvalues.

In $\mathbb{R}^n$, every basis of n vectors is complete, i.e., every vector in $\mathbb{R}^n$ can be expanded into a linear combination.

Therefore, the above procedure always works in finite-dimensional systems.

But PDEs are something closer to an infinite dimensional system of ODEs.

Since we cannot count infinitely many eigenvectors and compare them to the dimension of the vector space, it is **much** more complicated and subtle to understand the concept of **completeness** of eigenfunctions.

So we will try to understand when Fourier series methods work

$$\varphi(x) \overset{?}{=} \sum_{k=0}^{\infty} a_k u_k$$

Questions:

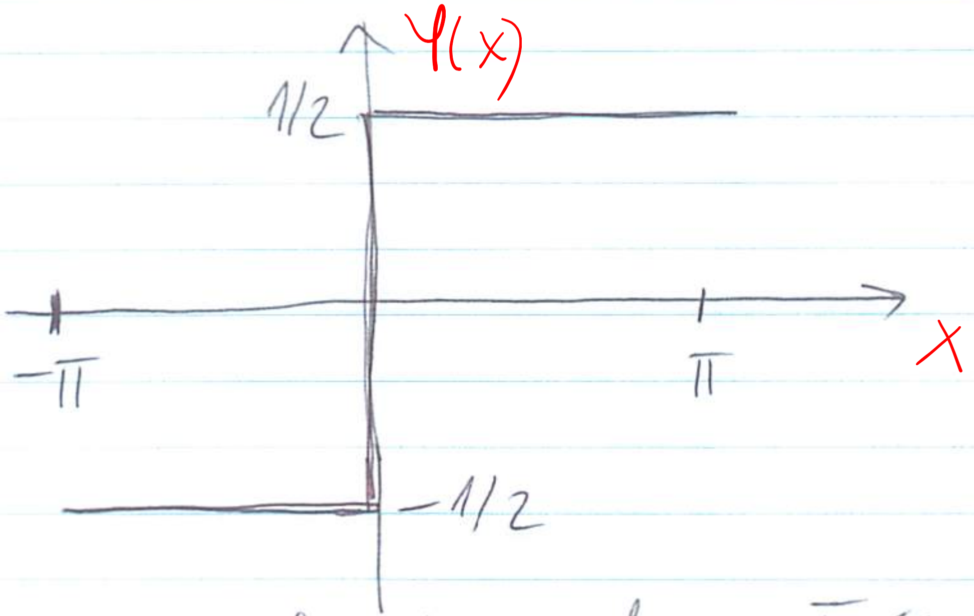
① When does the Fourier series of a function converge?

② If it does converge, in what sense does it converge and how fast?

Let's take a specific and very instructive sample to illustrate the ⑥

Gibbs Phenomenon

Let's take the function $f(x)$



$$f(x) = \begin{cases} -1/2 & \text{if} \quad -\pi < x < 0 \\ 1/2 & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x = 0 \end{cases}$$

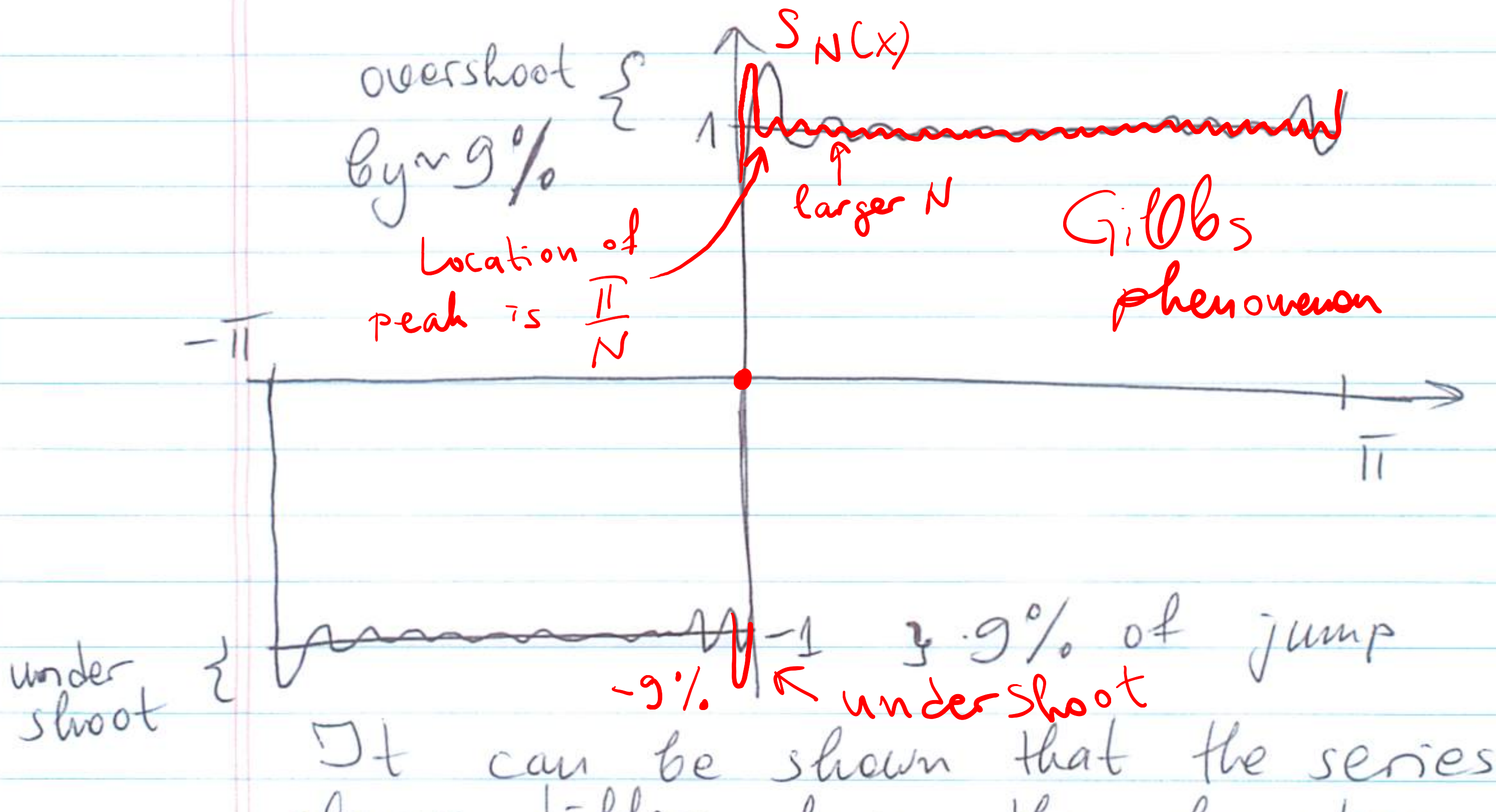
Fourier series

$$f(x) \overset{?}{=} \sum_{n \text{ odd}} \frac{2}{n\pi} \sin(nx)$$

If we truncate the series to a finite number of terms

$$\underset{\approx f(x)}{S_N(x)} = \sum_{\substack{n \text{ odd},\\ n \leq N}}^{N} \frac{2}{n\pi} \sin(nx)$$

and plot the result, we get:



It can be shown that the series always differs from the function near the discontinuity

$$\lim_{M\to\infty} S_M\Big(\underset{\swarrow \atop 0}{\frac{\pi}{M}}\Big) = \int_{-\pi}^{\pi} \frac{\sin\theta}{\theta}\frac{d\theta}{2\pi} \simeq 0.59$$

Funny facts:

① There exists an integrable function whose Fourier series diverges at every point

② There exists a continuous function whose series diverges at many points

These show us that we need to be careful and restrict ourselves to certain classes of functions (function spaces) in order to make any statements.

We won't prove the following theorems in class, rather, we will try to understand the different types of convergence:

uniform, pointwise, and in norm

Pointwise convergence

Let the truncated series be

$$S_N(x) = \sum_{|n| \leq N} a_k u_k$$

and denote the remainder (error, residual)

$$R_N = \varphi(x) - S_N(x)$$

The Fourier series converges pointwise if

$$\boxed{\lim_{N \to \infty} R_N(x) = 0 \qquad \forall x \in I}$$

That is, the series converges at every point.

But note that each x may require a different N to get a good approximation, that is, for some x the series may converge (much) slower/ faster.

To make the Fourier series solution useful in practice, however, we would like to be able to truncate it to a finite number of terms and get a good approximation of the solution everywhere!

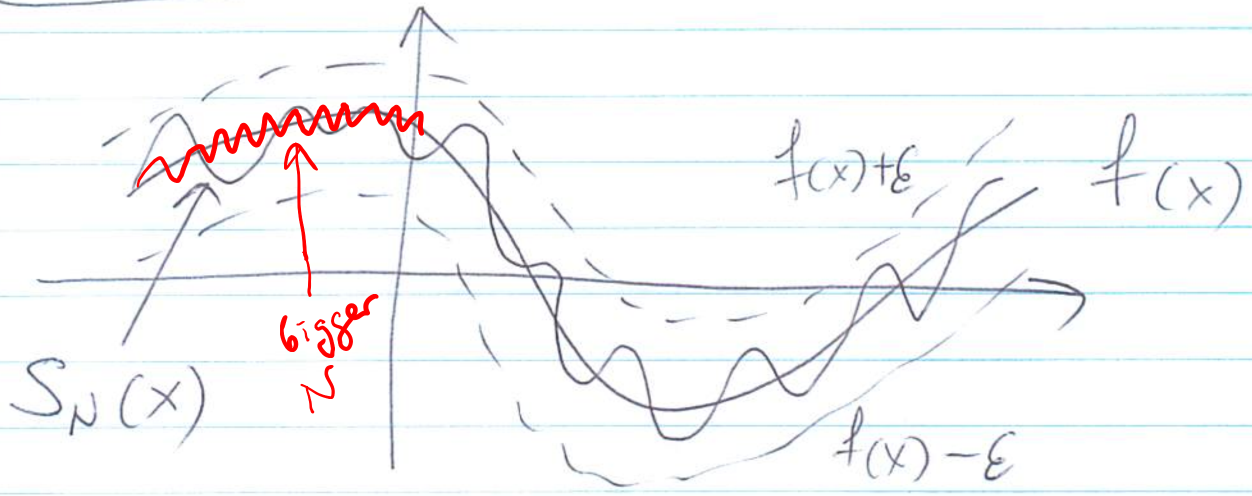
This requires

Uniform Convergence

$$\lim_{N\to\infty} \| R_N(x) \|_\infty = 0$$

i.e.

$$\boxed{\lim_{N\to\infty} \left\{ \max_{a \le x \le b} | R_N(x) | \right\} = 0}$$



The Gibbs phenomenon means that for the Heaviside (step) function (and all discontinuous functions) we get pointwise but not uniform convergence

Uniform $\Rightarrow$ pointwise
but not vice-verse

We now state some classical theorems without proofs:

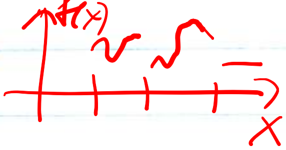
Uniform convergence of Fourier Series

If $f(x) \in C^2$ and satisfies the BCs, that is, if $f(x)$, $f'(x)$ and $f''(x)$ exist and are continuous on $[a,b]$ and $f(a) = f(b) = 0$ (for Dirichlet) then the Fourier series of $f(x)$ converges uniformly on $[a,b]$

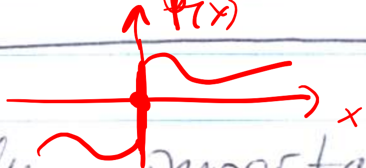
twice continuously differentiable

Pointwise convergence of FS

If $f(x)$ is a piecewise continuous on $[a,b]$ and $f'(x)$ is also piecewise continuous, then the Fourier series converges pointwise on (a,b),



$$\boxed{\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left[f(x^+)+f(x^-)\right]}$$



Is it really important that the Fourier series converge for every point ?

$u_t \underset{\uparrow}{=} \mathcal{L} u$
for every x ?

What if

$$\tilde{f}(x) = \lim_{N\to\infty} \sum c_n f_n(x)$$

differs from $f(x)$ at only a countable number of points ?
Would that be good enough in practice ?
Probably ...

So let's also consider

Convergence in norm / L_2

$$\lim_{N \to \infty} \| R_N(x) \|_2^2 = 0$$

i.e

$$\boxed{\lim_{N \to \infty} \int | R_N(x) |^2 \, dx = 0 \quad \neq \ R_N(x) = 0 \quad \forall x}$$

Observe that the integral does not see the value of $R_N(x)$ at a non-dense subset of $\mathbb{R}$, so this is definetely different than either pointwise or uniform convergence

$$\| u_t - \mathcal{L} u \|_2 = 0$$

$$\boxed{\text{uniform} \Rightarrow L_2}$$

There is no relationship between pointwise and L_2 convergence

L_2 convergence is also called <u>mean-square convergence</u>

Hilbert space of
square-integrable functions

Theorem: L_2 convergence of FS

If $f \in L_2$, which means

$$\|f\|_2^2 = \int_a^b |f(x)|^2 dx$$

is finite

then the Fourier series of $f(x)$ converges in L_2 norm

Theorem Parseval's equality

$f \in L_2$ iff

$$\|f\|_2^2 = \sum_{n=1}^{\infty} |A_n|^2 \, \|u_n\|_2^2$$

eigenfunctions

This means the norm ("power") of the function is contained in its Fourier series (no "power" is missed) note: sine, cos, complex exp are

Completeness

An infinite set of orthogonal functions

$\{ u_1(x), u_2(x), \ldots \}$

is complete if

① Parseval's equality is true for all $f \in L_2$

or, equivalently

② There is no "nontrivial" $f \in L_2$ that is orthogonal to all u_k's

Here "trivial" means that $f(x)$ is zero almost everywhere, i.e., it is nonzero on a set of "measure zero"

Any function $f(x) \in L^2$ can be expanded in a complete basis and the orthogonal series converges in the mean-square sense, and Parseval's equality holds.

So this applies more generally than Fourier series