# PDE Spring 2016 ①

# Lecture 13

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## Fourier Series part 1
(Review)

Before we begin Fourier series I would like to review some abstract but key concepts from linear algebra

A vector space $\mathcal{V}$ is a set of elements $\vec{x} \in \mathcal{V}$ such that functions

① Scalar multiplication

$$\alpha \vec{x} \in \mathcal{V}$$ function sp

② Vector addition

$$\vec{z} = \alpha \vec{x} + \beta \vec{y} \in \mathcal{V}$$

A subspace $\mathcal{V}' \subseteq \mathcal{V}$ is a subset that is closed under scalar multiplication and addition, i.e., under linear combinations.

E.g. $\mathbb{R}^3 \equiv \mathcal{V}$

$$\mathcal{V}' = \{ \vec{x} \in \mathbb{R}^3 \mid x_3 = 0 \} \sim \mathbb{R}^2$$

A set of vectors is <u>linearly independent</u> or forms a <u>basis</u> if there is no nontrivial linear combination equal to zero

$$\sum_{i=1}^{n} \alpha_i \vec{a}_i = 0 \Rightarrow \alpha_i = 0$$

The <u>dimension</u> of a vector (sub) space is the number of elements in a basis

$$\dim \mathcal{V} = n \quad \rightarrow \infty \text{ for function spaces}$$

e.g. $\dim \mathbb{R}^n = n$

Every $\vec{z} \in \mathcal{V}$ can be <u>uniquely</u> represented in a given basis

$$\vec{z} \overset{?}{=} \sum_{i=1}^{\cancel{n} \to \infty \text{ for us}} \alpha_i \vec{a}_i$$

That is, a vector can be represented as a list of $n$ coefficients in a given basis:

$$\vec{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

For $\mathbb{R}^n$ the usual assumed basis is

$$\alpha_1 = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \alpha_2 = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{etc.}$$

If we collect a set of $m$ vectors we get a <u>matrix</u>

$$A = \left[ a_1 \mid a_2 \mid \dots \mid a_m \right]$$

columns of $A$

A matrix-vector product is

$$b = Ax = \sum_{i=1}^{m} x_i a_i$$

Observe that $Ax$ is an element of some other vector space, called image or range or column space of matrix $A$

Therefore

$$y = Ax$$

defines a linear mapping or transformation (operator) from one (finite-dimensional) vector space to another.

The actual coefficients depend on the choice of basis (both for the domain and the image of the mapping) but the key concept is the mapping itself.

Linearity means that

Linear differential operator

$$\begin{cases} A(\alpha x) = \alpha A x \\ A(x+y) = Ax + Ay \end{cases}$$

as we already explained

Every linear transformation can be represented by a matrix domain column space/range/image

$$A : \mathcal{V} \rightarrow \mathcal{V}'$$

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The dimension of the column space of A is the **rank**

$$r = \text{rank}\, A = \dim \mathcal{V}'$$

$$\boxed{r \leq \min(m, n)}$$

does not make sense for function spaces

If $r = \min(m, n)$ the matrix is **full rank**

The **null space** of a matrix is the set of solutions of

$$Ax = 0$$

$$\text{null}\, A = \{ x \mid Ax = 0 \}$$

This is also a subspace $\mathcal{V}' \subseteq \mathcal{V}$ (**prove it**)

**nullity** $A = \dim(\text{null}\, A)$ is the number of linearly independent solutions of $Ax = 0$

# Fundamental theorem of LA:

$$\text{rank} + \text{nullity} = n$$

for $A \equiv [m \times n]$ matrix

i.e. $A \in \mathbb{R}^{m,n}$, or $\mathbb{C}^{m \times n}$

Most of the time we are interested in one vector space and mappings from it to itself, i.e., $n \times n$ square matrix

A square matrix forms a basis or is invertible iff

$$\exists B \equiv [n \times n] \text{ s.t. } AB = BA = I$$

where $I$ is the identity matrix

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$$

Kronecker $\delta$

$$B \equiv A^{-1}$$

$$\mathcal{L}u = f$$

$$u = \mathcal{L}^{-1} f$$

The following are equivalent: 7

1. A is invertible
2. A is full-rank, rank $A = n$
3. Columns of A are a basis
4. $\det A \neq 0$
5. Zero is not an eigenvalue of A, i.e., there is no nonzero solution of

$$Ax = 0$$

PDEs

Expressing one vector in a given basis means solving the linear system

$$\underset{\substack{\uparrow \\ \text{coefficients}}}{Ax} = y \leftarrow \text{vector}$$

If A is invertible a solution always exists,

$$x = A^{-1} y$$

$$u = \mathcal{L}^{-1} f$$

# Orthogonality

A combination of a vector space with an inner or dot product gives an inner-product space.

Inner product is a map from two elements of $V$ to a scalar (element of a field)

$$(x, y) \equiv \langle x, y \rangle \equiv x \cdot y$$

$$(x, y) : V \times V \to \mathbb{C}$$

and satisfies: the three axioms.

1. $(x, y) = \overline{(y, x)}$ conjugacy or symmetry

2. Linearity in first argument

$$(\alpha x, y) = \alpha (x, y)$$

$$(x + y, z) = (x, z) + (y, z)$$

(3) Positive definiteness

$(x, x) \geq 0$ and real

and $(x, x) = 0$ iff $x = 0$

<u>Standard</u> inner product

$$(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

Two vectors are <u>orthogonal</u> if

$$(\vec{x}, \vec{y}) = 0$$

real $\|\vec{x}\|^2 = (\vec{x}, \vec{x}) = \vec{x} \cdot \vec{x} \geq 0$

$L_2$ norm or <u>length</u> of vector, better denoted with $\|\vec{x}\|_2^2$

$$\boxed{\|\vec{x}\|_2 = (\vec{x}, \vec{x})^{1/2}}$$

A vector is <u>normalized</u> if

$$\|\vec{x}\| = 1$$

An adjoint of a linear operator is defined by (10)

$$B = A^* \text{ if}$$

$$\boxed{(Ax, y) = (x, A^* y)}$$ ✓

For finite dimensional matrices, this is just the conjugate transpose for complex matrices

$$A^* = \bar{A}^T$$ ← ~~transpose~~ for infinite dimensional operators

or for real matrices just the transpose.

But the above definition is the more general one.

Two vector spaces are orthogonal if

$$\mathcal{V} \perp \mathcal{V}' \text{ if } \forall x \in \mathcal{V}, \forall y \in \mathcal{V}', \ (x,y) = 0$$

E.g. $\{z \in \mathbb{R}^3 \mid z_1 = 0, z_2 = 0\}$

and $\{z \in \mathbb{R}^3 \mid z_3 = 0\}$

A matrix is <u>unitary</u> if its columns are normalized and orthogonal to each other.

This implies

$$\boxed{U^{-1} = U^{*}}$$

For real matrices usually the word <u>orthogonal</u> matrix is used.

Recall the problem of expressing a vector $y$ in an <u>orthonormal</u> basis given by the columns of $U$:

$$U\vec{x} = \vec{y} \Rightarrow$$

$$\vec{x} = U^{-1}\vec{y} = U^{*}\vec{y}$$

$$\Rightarrow \boxed{x_i = (\vec{y}, \vec{u}_i)}$$

orthogonal projection onto each column of $U$.

How to see this quickly

$$\vec{y} = \sum_i x_i \vec{u}_i$$

Dot both sides by $\vec{u}_j$

$$\vec{y} \cdot \vec{u}_j = \sum_i x_i (\vec{u}_i \cdot \vec{u}_j)$$

$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$$= \sum_i x_i \delta_{ij} = x_j$$

$$\Rightarrow \quad x_j = \vec{y} \cdot \vec{u}_j$$

as we claimed.

More general vector norm:

$L_p$ norm: $\|X\|_p = \left( \sum_{i=1}^{N} |x_i|^p \right)^{1/p}, \; p \geq 1$

① $p=1$, $L_1$ norm

$$\|X\|_1 = \sum_i |x_i|$$

② $p=2$, $L_2$ norm

$$\|X_2\| = (x,x)^{1/2} = \sqrt{\sum_{i=1}^{N} x_i^2}$$

③ $p=\infty$, $\|X\|_\infty = \max_i |x_i|$

A norm (vector norm) is itself a well-defined mathematical concept. (13)

It is a mapping from $V \to \mathbb{R}$
vectors to real numbers

axioms
$$\begin{cases} (1)\ \ \|\alpha \vec{v}\| = |\alpha|\, \|\vec{v}\| \\ (2)\ \ \|\vec{v}+\vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\| \\ \qquad \text{(triangle inequality)} \\ (3)\ \ \text{If } \|\vec{v}\| = 0 \iff \vec{v} = 0 \end{cases}$$

These imply $\begin{cases} \|\vec{v}\| \geq 0 \\ \text{non-negativity} \end{cases}$

So we can think of the norm as a generalization of length of a vector.

# Eigenvalues

For every square matrix (14)
$\exists$ at least one $\lambda$ s.t.

$$\underset{\substack{\uparrow\\ \text{eigenvector}}}{A\,x} = \lambda\, x \quad , \quad x \neq 0$$

eigenvalue

A matrix is non-defective or diagonalizable if it has $n$ linearly independent eigenvectors.

Form $X = [\vec{x}_1 \mid \vec{x}_2 \mid \dots \mid \vec{x}_n]$

and $X$ is invertible.

Then

$$A X = X \Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

is a diagonal matrix of eigenvalues.

This means

$$A X X^{-1} = X \Lambda X X^{-1}$$ Multiply on right by $X^{-1}$

$$A I = A = X \Lambda X^{-1}$$

$$\boxed{A = X \Lambda X^{-1}}$$

is the eigenvalue decomposition

If we want to multiply a vector by the matrix:

$$A y = X \Lambda (X^{-1} y) = X \Lambda \tilde{y}$$

Observe that

$$X^{-1} y = \tilde{y} \Rightarrow y = X \tilde{y}$$

that is, $\tilde{y}$ is just $y$ expressed in the basis of eigenvectors.

$$z_i = (\Lambda \tilde{y})_i = \lambda_i \tilde{y}_i \quad (\vec{z} = \Lambda \tilde{y})$$

$$X \vec{z} = A \vec{y}$$

so that in this basis the matrix is <u>diagonal</u>

Now $Ay = Xz$ is simply $z$ expressed back in the original basis.

The eigenvectors give us a special <u>basis</u> in which the matrix (linear transformation) is <u>diagonal</u>
(this is the key concept)

A matrix is <u>normal</u> if

$$AA^* = A^*A$$

Theorem: $A$ is normal iff (unitarily diagonalizable) one can find an <u>orthonormal set</u> of eigenvectors that forms a basis:

$$\boxed{A = U \Lambda U^{-1} = U \Lambda U^*}$$

unitary matrix

$$A^{-1} = (U \Lambda U^*)^{-1} = (U^*)^{-1} \Lambda^{-1} U^{-1} = U \Lambda^{-1} U^*$$

$$A = U \Lambda U^*$$

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is the best representation of a normal matrix for many purposes.

A matrix is <u>Hermitian</u> if or <u>self-adjoint</u>

$$A^* = A,$$

called <u>symmetric</u> in the real case,

$$A^T = A$$

Since $A^* A = A^2 = A A^*$ the matrix is also normal and so

$$\boxed{\text{Every hermitian matrix is unitarily diagonalizable and all its eigenvalues are real}}$$

We will see how to generalize these properties to functions (as vectors) and linear operators (as matrices) next time

Theorem: Eigenvectors corresponding to distinct eigenvalues of a Hermitian / symmetric matrix are orthogonal $(A = A^*)$

Proof: $\begin{cases} A x_1 = \lambda_1 x_1 \\ A x_2 = \lambda_2 x_2 \end{cases}$

$$(x_1, A x_2) = (x_1, \lambda_2 x_2) = \lambda_2 (x_1, x_2)$$

$$= (A^* x_1, x_2) = (A x_1, x_2) = (\lambda_1 x_1, x_2)$$

$$= \lambda_1 (x_1, x_2)$$

$$\Rightarrow (\lambda_1 - \lambda_2)(x_1, x_2) = 0$$

Since $\lambda_1 \neq \lambda_2 \Rightarrow (x_1, x_2) = 0$

as needed to show

Note: It can also be shown that eigenvectors corresponding to the same eigenvalue can be made orthogonal (Gram-Schmidt procedure)

Similar proofs can be done to show that eigenvalues are real:

Recall adjoint property of dot product:

$$\overline{(x,y)} = (y,x)$$

$$\Rightarrow \overline{(Ax,x)} = (Ax,x)$$

$$\overline{\lambda(x,x)} = \lambda(x,x)$$

Since $(x,x) > 0$ and real (recall $(x,x) = \|x\|_2^2$), we get

$$(\overline{\lambda} - \lambda)(x,x) = 0 \Rightarrow$$

$$\overline{\lambda} = \lambda \text{, i.e., } \lambda \text{ is real}$$

<u>Conclusion</u>: Every Hermitian matrix is unitarily diagonalizable with <u>real eigenvalues</u> <u>and</u> <u>real eigenvectors</u> (complex numbers not necessary)