PDE Spring 2016

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Lecture 12

Separation of Variables

In this first lecture of the second half of the semester we will motivate the rest by looking at the heat equation on a <u>bounded domain</u>:

$$
\begin{cases}
u_t = k u_{xx} \\
0 < x < 1, \quad t > 0 \\
u(x,0) = g(x) & \text{IC} \\
u(0,t) = u(1,t) = 0 & \text{homogeneous Dirichlet BC}
\end{cases}
$$

Where we do <u>not</u> require that $g(1) = g(0) \neq 0$, allowing the solution to start discontinu

One way to solve this would be to find the Green's function by setting

$$g(x) = \delta(x - x_0)$$

But this is almost as hard as the original problem because of the <u>nontrivial BCs</u>.

Let us try the <u>ansatz</u> (educated guess)

$$u(x,t) = \underbrace{X(x)}_{\text{function only of } x}\ \underbrace{T(t)}_{\text{Function only of } t}$$

This means the two independent <u>variables are separated</u>.

It will take us several lectures to understand why this was a good guess...

But for now let's try it

BCs give:

$$\begin{cases} u(0,t) = X(0)\,T(t) = 0 \\ u(1,t) = X(1)\,T(t) = 0 \end{cases}$$

This means (since $T \neq 0$)

$$\boxed{X(0) = X(1) = 0}$$

i.e. the spatial part satisfies the BCs

Now, plug into PDE

$$u_t = X(x)\,T_t(t) = k\,X_{xx}\,T$$

$$k\,u_{xx} = k\,X_{xx}(x)\,T(t)$$

$$\Rightarrow \quad \underbrace{\frac{1}{kT(t)}\,T'(t)}_{\text{Function of } t} = \underbrace{\frac{1}{X(x)}\,X''(x)}_{\text{Function of } x}$$

Since two functions of two different variables are equal, they must both be constant:

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$$\boxed{\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda}$$

Sign for later $\lambda > 0$

This is now a system of two ODEs!

The easier one to solve is

$$T' = -\lambda k T \Rightarrow T(t) \text{ unknown}$$

$$\boxed{T(t) = T(0)\exp(-\lambda k t)}$$

$\lambda > 0$, $k > 0$

and this exponential decay is typical of the heat equation

The harder equation is

$$\text{BVP}\begin{cases} X'' = -\lambda X \\ X(0) = X(1) = 0 \end{cases}$$

This is called a <u>two-point boundary value problem</u>

The trivial solution is (5)

$\bar{X}(x) = 0$

but we want nontrivial ones.

The general solution has the form

$$\bar{X} = a e^{\alpha x}$$

$\alpha = \pm\sqrt{-\lambda}$

where $\alpha^2 = -\lambda$ from the ODE

Since the sign of α is arbitrary

$$\bar{X} = a e^{\alpha x} + b e^{-\alpha x}$$ $x e^{\alpha x}$

is the general solution (we need two constants for second-order)

$\bar{X}(0) = a + b = 0$ $\Rightarrow a = -b$

$\bar{X}(1) = a e^{\alpha} + b e^{-\alpha} = 0$

$$= a(e^{\alpha} - e^{-\alpha}) = 0$$

If $a = 0 \Rightarrow b = 0$ so trivial

If $a \neq 0$ then

$$e^{\alpha} - \frac{1}{e^{\alpha}} = 0 \Rightarrow (e^{\alpha})^2 = 1$$

Now let's assume that

$\lambda < 0$

$\alpha^2 = -\lambda > 0$ and $\alpha > 0$ is <u>real</u>

$e^{\alpha} = +1 \Rightarrow \alpha = 0$

which is a contradiction

If $\lambda = 0$ then equation is

$$X'' = 0 \Rightarrow X(x) = ax + b$$

and $X(0) = X(1) = 0$ forces $a = b = 0$

so we get the trivial solution

But note this is a <u>particular solution</u> if $X(0)=0$, $X(1)=1$ nonhomog. BCs

We conclude

$\boxed{\lambda > 0}$

$\alpha = \sqrt{-\lambda} = i\beta \Rightarrow -\lambda = -\beta^2$

$\alpha^2 = -\lambda < 0$ so $\alpha = i\beta$ is <u>complex</u>

$\beta = \sqrt{+\lambda}$

$$e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$$

and so it is better to switch now to sines and cosines

Neumann: $X'(1) = 0$ would have both sin, cos

$$X(x) = A\cos\beta x + B\sin\beta x$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(1) = B\sin(\beta\cdot 1) = 0 \Rightarrow$$

$$\sin(\beta) = 0 \Rightarrow X(x) = B\sin(n\pi x)$$

since $B = 0$ is trivial solution.

$$\Rightarrow \quad \beta = (\text{integer, positive})\,\pi$$

$$\beta = n\pi \qquad n \in \mathbb{Z}^+$$

$$\alpha^2 = -\beta^2 = -\lambda \Rightarrow \qquad \sin(-\beta x) = -\sin(\beta x)$$

$$\boxed{\lambda = n^2\pi^2} \quad \text{where } n \in \mathbb{Z}^+$$

This is the fundamental result of the separation of variables:

The set of solutions of

$\partial_x(a(x)u_x) \longrightarrow \mathcal{L}u = 0 \quad \mathcal{B}u = 0$

$\mathcal{L}$ independent of t

is countable and can be eigenfunctions with $T = 0$ enumerated by positive integers.

$$Ax = 0 \Rightarrow Ax = \lambda x, \lambda = 0$$

Sidenote: Compare this to the case of finite-dimensional

$$Ax = 0$$

which has a finite number of independent solutions, called the nullity of the matrix

$$X(x) = B \sin(n\pi x)$$

$$\Rightarrow \boxed{u_n(x,t) = B_n e^{\substack{-kn^2\pi^2 t \\ -k\lambda t}} \sin(n\pi x)}$$

We have therefore identified a family of countably infinetely many solutions.

Claim: Any solution can be be expressed as a sum of these

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t} \sin(n\pi x)$$

How to determine B_n?

The one thing we have not used is the initial condition:

$$u(x,0) = \boxed{\sum_{n=1}^{\infty} B_n \sin(\pi n x) = g(x)}$$

The question now becomes:

For what functions $g(x)$ does a convergent series of the above form exist? When it exists, is it unique, and if so, how to determine the B_n coefficients?

To answer this question we will need to go into <u>Fourier analysis</u>

It turns out almost every function can be expanded in this way and

$$B_n = 2 \int_{x=0}^{1} g(x) \sin(n\pi x)\, dx$$

as we will obtain later.

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$\mathcal{L} = \partial_{xx}$

Some comments:
the problem $\rightarrow$ $\mathcal{L}\underline{\bar{X}} = -\lambda \underline{\bar{X}}$

$$\underline{\bar{X}}'' = -\lambda \underline{\bar{X}} \quad , \quad \underline{\bar{X}}(0) = \underline{\bar{X}}(1) = 0$$

is an <u>eigenvalue problem</u> very similar to the matrix $Ax = \lambda x$

The λ's that are possible are the <u>eigenvalues</u> and $\lambda = n^2\pi^2$

$$\underline{\bar{X}}(x) = \sin(n\pi x)$$

are the <u>eigenfunctions</u>.

Many of the concepts from linear algebra will generalize, so we will review some of those in the upcoming lectures.

Also observe that the <u>high-frequency</u> modes (large n) <u>decay very rapidly</u> $(\exp(-kn^2\pi^2 t))$ and as t grows only the small n's will survive and as $t \to \infty$ the mode $n=0$ dominates.

$\rightarrow$ SMOOTHING PROPERTY

Also try: $\begin{cases} u(0,t)=0 \\ u_x(1,t)=0 \end{cases}$

Exercise for home

Change the BCs to <u>homogeneous Neumann</u>

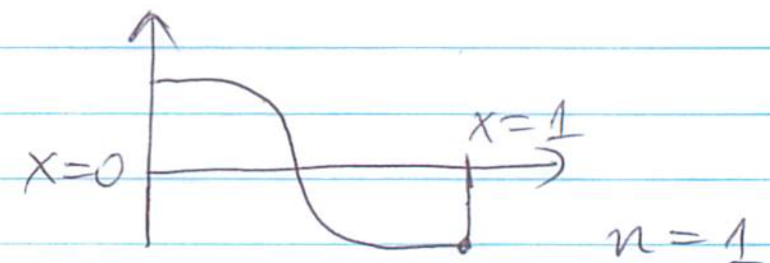
$$u_x(0,t) = u_x(1,t) = 0$$

and show that the separable solution has the form

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \cos(\pi n x)$$

this is $n=0$

satisfies BC



Now the IC becomes the equation for A_0 and A_n:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = g(x)$$

Note that the same derivation can be generalized to the wave equation (and others)