Lecture 12

Separation of Variables

In this first lecture of the second half of the semester, we will motivate the rest by looking at the heat equation on a bounded domain:

\[
\begin{align*}
    u_t &= k u_{xx} \\
    0 < x < 1, \quad t > 0 \\
    u(x, 0) &= g(x) \quad \text{IC} \\
    u(0, t) &= u(1, t) = 0 \quad \text{homogeneous Dirichlet BC}
\end{align*}
\]

Where we do not require that \( g(1) = g(0) \neq 0 \), allowing the solution to start discontinuous.
One way to solve this would be to find the Green's function by setting
\[ g(x) = \delta(x - x_0) \]
But this is almost as hard as the original problem because of the nontrivial BCs.

Let us try the ansatz (educated guess)

\[ u(x, t) = \overline{X}(x) \overline{T}(t) \]

This means the two independent variables are separated.

It will take us several lectures to understand why this was a good guess...

But for now let's try it
BCs give:
\[
\begin{align*}
    u(0, t) &= \bar{x}(0) T(t) = 0 \\
    u(1, t) &= \bar{x}(1) T(t) = 0
\end{align*}
\]
This means (since \( T \neq 0 \))
\[
\bar{x}(0) = \bar{x}(1) = 0
\]
i.e. the spatial part satisfies the BCs

Now, plug into PDE
\[
\begin{align*}
    u_t &= \bar{x}(x) T_t(t) = k \bar{x}_x T(t) \\
    k u_{xx} &= k \bar{x}_{xx}(x) T(t)
\end{align*}
\]
\[
\Rightarrow \quad \frac{1}{k T(t)} T'(t) = \frac{1}{\bar{x}(x)} \bar{x}''(x)
\]
Function of \( t \) \quad Function of \( x \)

Since two functions of two different variables are equal, they must both be constant.
\[ \frac{1}{k} \frac{T'}{T} = \frac{X''}{X} = -\lambda \]

Sign for later \( \lambda > 0 \)

This is now a system of two ODEs!

The easier one to solve is

\[ T' = -\lambda k T \Rightarrow T(t) \text{ unknown} \]

\[ T(t) = T(0) \exp(-\lambda k t) \]

This exponential decay is typical of the heat equation.

The harder equation is

\[ \begin{cases} X'' = -\lambda X \\ X(0) = X(1) = 0 \end{cases} \]

This is called a two-point boundary value problem.
The trivial solution is 

\[ X(x) = 0 \]

but we want nontrivial ones.

The general solution has the form

\[ X = ae^x \]

where \( \lambda^2 = -\lambda \) from the ODE

Since the sign of \( a \) is arbitrary

\[ X = ae^x + be^{-x} \]

is the general solution (we need two constants for second-order)

\[ X(0) = a + b = 0 \Rightarrow a = -b \]

\[ X(1) = ae + be^{-1} = 0 \]

\[ = a(e^1 - e^{-1}) = 0 \]

\( \Rightarrow a = 0 \Rightarrow b = 0 \) so trivial

\( \text{If } a \neq 0 \text{ then} \)

\[ e^x - \frac{1}{e^x} = 0 \Rightarrow (e^x)^2 = 1 \]
New let's assume that

\[ \lambda < 0 \]

\[ \lambda^2 = -\lambda > 0 \text{ and } \lambda > 0 \text{ is real} \]

\[ e^\lambda + 1 \Rightarrow \lambda = 0 \]

which is a contradiction

If \( \lambda = 0 \) then equation is

\[ x'' = 0 \Rightarrow x(x) = ax + b \]

and \( x(0) = x(1) \geq 0 \) forces \( a = b = 0 \)

so we get the trivial solution

\[ \begin{aligned}
\text{We conclude} & & \lambda > 0 \\
\lambda & = \sqrt{-\lambda} = i\beta \Rightarrow \lambda - \lambda = -\beta^2
\end{aligned} \]

\[ \lambda^2 = -\lambda < 0 \text{ so } \lambda = i\beta \text{ is complex} \]

\[ e^{i\beta x} = \cos(\beta x) + i\sin(\beta x) \]

and so it is better to switch now to sines and cosines
\[ X'(1) = 0 \text{ would have both } \sin, \cos \]

\[ X(x) = A \cos \beta x + B \sin \beta x \]

\[ X(0) = 0 \Rightarrow A = 0 \]

\[ X(1) = B \sin (\beta 1) = 0 \Rightarrow \sin (\beta 1) = 0 \Rightarrow X(x) = B \sin (n \pi x) \]

since \( B = 0 \) is trivial solution.

\[ \Rightarrow \beta = \left( \text{integer, positive} \right) \Rightarrow \beta = n \pi \quad n \in \mathbb{Z}^+ \]

\[ \lambda = n \pi^2 \]

This is the fundamental result of the separation of variables.

The set of solutions of

\[ \Delta u = 0 \quad \Delta u = 0 \]

is independent of \( t \) and can be enumerated by positive integers.
\[ Ax = 0 \Rightarrow Ax = \lambda x, \lambda \neq 0 \]

\[ \text{Side note: Compare this to the case of finite-dimensional } Ax = 0 \]

Which has a finite number of independent solutions, called the nullity of the matrix.

\[ \bar{x}(x) = B \sin (\pi \bar{x}) \]

\[ \Rightarrow \]

\[ u(x,t) = B_n e^{-k \pi^2 t} \sin (n \pi x) \]

We have therefore identified a family of countably infinitely many solutions.

\[ \text{Claim: Any solution can be expressed as a sum of these} \]

\[ u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k \pi^2 t} \sin (n \pi x) \]

How to determine \( B_n \)?
The one thing we have not used is the initial condition:

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin (\pi n x) = g(x)$$

The question now becomes:

For what functions $g(x)$ does a convergent series of the above form exist? When it exists, is it unique, and if so, how to determine the $B_n$ coefficients?

To answer this question we will need to go into Fourier analysis.

It turns out almost every function can be expanded in this way and

$$B_n = 2 \int_{x=0}^{1} g(x) \sin (\pi n x) \, dx$$

as we will obtain later.
Some comments:

The problem

\[ \ddot{x} = -\lambda \dot{x}, \quad \dot{x}(0) = \dot{x}(1) = 0 \]

is an eigenvalue problem very similar to the matrix \( AX = \lambda X \).

The \( \lambda \)'s that are possible are the eigenvalues and

\[ \lambda(x) = \sin(\pi n x) \]

are the eigenfunctions.

Many of the concepts from linear algebra will generalize, so we will review some of those in the upcoming lectures.

Also observe that the high-frequency modes (large \( n \)) decay very rapidly (\( \exp(-k^2 \pi^2 t) \)) and as \( t \) grows only the small \( n \)'s will survive and as \( t \to \infty \) the mode \( n = 0 \) dominates.
Also try: \[ \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \]

Exercise for home

Change the BCs to homogeneous Neumann

\[ u_x(0,t) = u_x(1,t) = 0 \]

and show that the separable solution has the form

\[ u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \cos(n\pi x) \]

This is \( n=0 \) satisfies BC

Now the IC becomes the equation for \( A_0 \) and \( A_n \):

\[ \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = g(x) \]

Note that the same derivation can be generalized to the wave equation (and others)