PDE Spring 2016

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Lecture 11

Green's Functions Revisited

Recall our solution formula for the heat equation

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, \varphi(y)\, dy$$

where in one dimension

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-x^2/(4kt)}$$

How do we understand this formula?

We are writing the solution as an integral of $G(x,t)$, which is itself a solution of the heat equation

So this must be the <u>superposition principle</u> @ play

What initial condition does the solution $G(x,t)$ satisfy? That is, what is (2)

$$\delta(x) = \lim_{t \to 0^+} G(x,t) \quad ?$$

If $x \neq 0$, then

$$\lim_{t \to 0^+} \frac{e^{-\alpha/t}}{\sqrt{t}} = 0 \quad \text{if } \alpha > 0$$

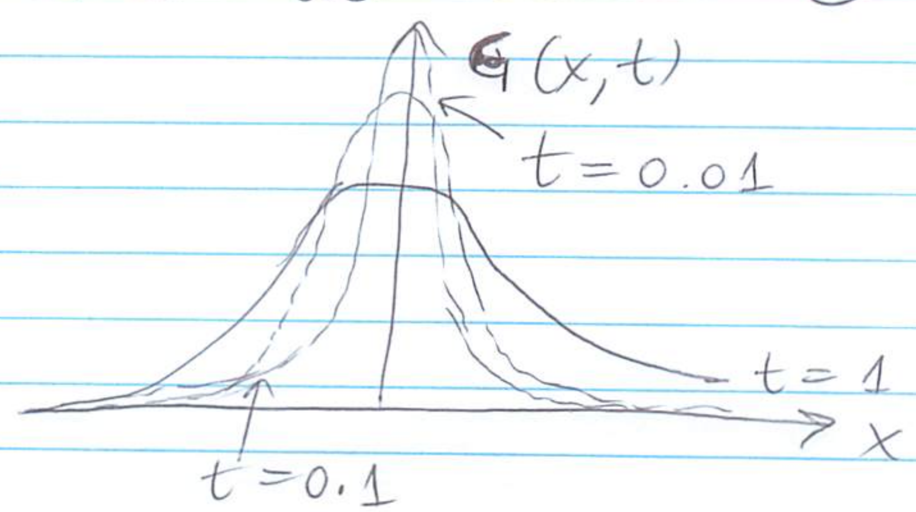
(e.g. plot this function near $t=0$)

So $\delta(x \neq 0) = 0$

But what about $x=0$?

$$\delta(x=0) \to \infty$$

since we have $e^{-0}/\sqrt{0} = 1/\sqrt{0}$



So as $t \to 0^+$ the Green's function becomes more and more peaked at the origin (3)

Remember that

$$\int_{x=-\infty}^{\infty} G(x,t)\, dx = 1 \qquad \forall t \geq 0$$

So we have

$$\begin{cases} \delta(x) = 0 \quad \text{for } x \neq 0 \\ \text{but} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1 \end{cases}$$

These two contradict each other if $\delta(x)$ is a function and the integral is the usual Riemann integral, since

$\delta(x) = 0$ for $x \neq 0$ <u>implies</u>

$\int \delta(x)\, dx = 0$

This means that $\delta(x)$ is <u>not a function</u> in the "classical" sense.

We will call it a <u>generalized function</u>

or a <u>distribution</u>

or a <u>functional</u>

or, most commonly, but in abuse of notation

the $\boxed{\delta \text{ (delta) "function"}}$

The physical intuition is that $G(x,t)$ is the solution of the heat equation starting from a <u>unit amount of "mass"</u> (quantity of interest) placed at $x=0$ at $t=0$

$\longrightarrow$ <u>unit point mass</u>

(common concept in physics but tricky mathematically)

If $\begin{cases} u_t = k\,u_{xx} \\ u(x,0) = \delta(x) \end{cases}$

then $u \equiv G(x,t)$

This is the most general <u>definition</u> of the Green's function. Solve

$$\begin{cases} \mathcal{L}u = 0 \\ u(x,0) = \delta(x) \end{cases}$$

so we can use this to define a Green's function for the heat, wave, and other equations.

But what is $\delta(x)$ really?

<u>Sidenote:</u>

Note that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} G(x,t)\,dx \neq \int_{-\infty}^{\infty} \lim_{t\to 0^+} G(x,t)\,dx$$

in the classical setting $\Rightarrow$ the integral and limit do <u>not commute</u>

We have avoided this by defining a δ "function"

What is

$$I = \int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = ?$$

if $f(x)$ is continuous

Since

$$I = \lim_{t \to 0^+} \int \frac{e^{-x^2/4kt}}{\sqrt{4kt}}\, f(x)\, dx$$

we expect only the value of f around $x=0$ to matter since this is where the δ function is peaked.

$$I = \int \delta(x)\, f(x)\, dx = f(0) \int \delta(x)\, dx$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)}$$

This is in fact a proper definition of the δ function.

A <u>functional</u> is a mapping from a function to a scalar, i.e., a scalar-valued function of a function (operator)

Consider a smooth, specifically continuous function $g(x)$.
This defines an operator or <u>functional</u>

$$\mathcal{L}_g[f(\cdot)] = \int_{x=-\infty}^{\infty} f(x)\,g(x)\,dx$$

The δ function is a <u>functional</u> that maps a function to its value at the origin

$$\boxed{\delta[f(\cdot)] = f(0) \equiv \int f(x)\,\delta(x)\,dx}$$

Since this is a <u>linear functional</u> we can apply <u>linear transformations</u>

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-y)\,dx = f(y)}$$

This is an "operational" definition of the δ function

So

$$\varphi(x) = \int_{-\infty}^{\infty} \delta(x-y)\,\varphi(y)\,dy$$

i.e, we have written our initial condition

$$u(x,0) = \varphi(x)$$

as a sum (integral)

The superposition principle says that the solution will be the same sum of the solution starting from initial condition $\delta(x-y)$.

But this is exactly the Green's function $G(x-y,t)$. So

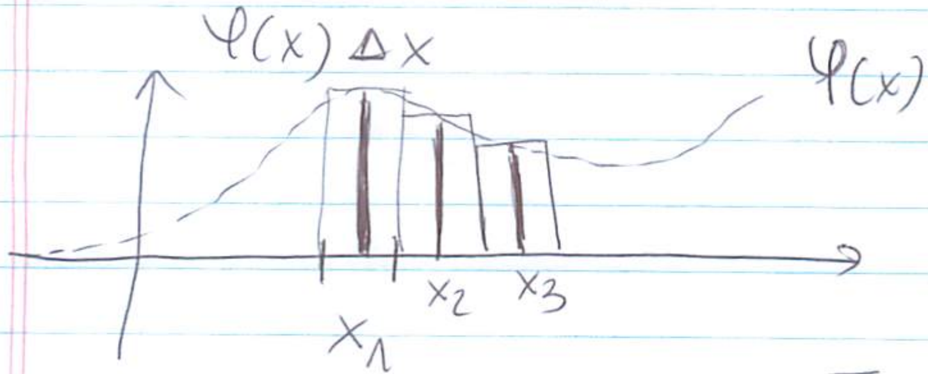
$$u(x,t) = \int G(x-y,t)\,\varphi(y)\,dy$$

as we concluded earlier.

The physical idea here is to split the initial condition into a sum of point masses:

total mass $= \varphi(x)\Delta x$

As Δx becomes smaller we get a δ function of total mass



Replace $\varphi(x)$ by $\Delta x \sum_i \varphi(x_i)\,\delta(x-x_i)$

$$\varphi(x) \longleftarrow \Delta x \sum_i \varphi(x_i)\,\delta(x-x_i)$$

$$\Rightarrow u(x,t) \longleftarrow \Delta x \sum_i G(x_i-x,t)\,\varphi(x_i)$$

As $\Delta x \to 0$ the sum becomes an integral

$$u(x,t) = \int G(y-x,t)\,\varphi(y)\,dy$$

which is the same as before.

The same intuition can be used to obtain Duhamel's principle

$$u_t - k u_{xx} = f(x,t)$$

To do this, we observe that $G(x,t)$ is <u>also</u> the solution to

$$\begin{cases} u_t - k u_{xx} = \delta(x)\,\delta(t) \\ u(x,-\infty) = 0 \end{cases}$$

What this means is that one cannot tell the difference, for $t>0$, between starting with unit point mass at $x=0$, or someone adding (source) a unit mass at $t=0$ to $x=0$

In other words, the solution of

$$\begin{cases} u_t = \delta(t) \\ u(-\infty) = 0 \end{cases}, \text{ where } u \equiv u(t)$$

is $\begin{cases} u = 0 \text{ for } t < 0 \\ u = 1 \text{ for } t > 0 \end{cases}$

Since $\displaystyle\int_{-\infty}^{x>0} \delta(s)\,ds = 1$

In fact recall that we derived G by differentiating the solution for an initial condition that is a step function.

For $t > 0$, the solution will be the same as for

$$\begin{cases} u_t = 0 \\ u(0) = 1 \end{cases}$$

Since in either case you just start with unity at $t \to 0^+$

Going back to

$$u_t - k u_{xx} = f(x,t)$$

write

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,\tau)\,\delta(x-s)\,\delta(t-\tau)\,ds\,d\tau$$

$\Rightarrow$ by superposition

$$u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-s,\,t-\tau)\,f(s,\tau)\,ds\,d\tau$$

But recall that $G(x,t)$ was only defined for $t \geq 0$. So let us define

$$G(x,t) = 0 \quad \text{for } t < 0$$

$\rightarrow$ this is a consequence of causality – the future cannot affect the present. So

$$u(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} G(x-s,\,t-\tau)\,f(s,\tau)\,ds\,d\tau$$

which is Duhamel's solution!

the physical picture is again the same: We decompose our forcing as a sum of unit or point forcing functions, and superimpose the solutions.

Key point is that once you know the response (solution) of a <u>linear</u> PDE to a unit source (point source), you can get the rest by superposition (integration)

The mathematics is however tricky. The δ function is not a real function and must be treated with care!

Specifically, only linear transformation of $\delta(x)$ are meaningful!

$\delta^2(x)$ or $\sqrt{\delta(x)}$

do <u>not</u> make sense

E.g.

$$h(x) = 2\delta(x) - 3\,\delta(x-1)$$

is a linear combination and makes sense

$$\int h(x)\, f(x)\, dx = 2\int \delta(x)\, f(x)\, dx$$

$$-3\int \delta(x-1)\, f(x)\, dx =$$

$$= 2f(0) - 3f(1)$$

So the functional here is

$$\mathcal{L}[f(\cdot)] = 2f(0) - 3f(1)$$

which is again linear

Similarly, integration is OK,

$$\int_a^x \delta(t-s)\, dt = \sigma(x-s)$$

step or Heaviside function

$$\sigma(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and for continuity $\delta(0) = 1/2$

Differentiation gives

$$\frac{d\sigma}{dx} \equiv \delta$$

which is new mathematics - we are new able to differentiate discontinuous functions

E.g.

$$\frac{d}{dx}|x| = \text{sign } x = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

so $$\frac{d^2}{dx^2}|x| = 2\delta(x)$$

Differentiation is also a linear operation so it is ok.

$\delta'(x)$ is also a distribution (linear functional)

$$\int \delta'(x) u(x)\, dx = \text{integrate by parts}$$

$$= -\int \delta(x) u'(x)\, dx + \Big[\delta(x) u(x)\Big]_{x=-\infty}^{\infty}$$

$$= -\int \delta(x) u'(x)\, dx = -u'(0)$$

So the derivative of $\delta(x)$ 16
is defining / is defined by
the functional

$$\mathcal{L}[f(\cdot)] = -f'(0)$$

which now acts on the
(vector) space of <u>continuously
differentiable</u> (test) functions.

There is a well-developed theory
of distributions. It mimics
much of linear algebra and
linearity is key → superposition