PDE Spring 2016

A. DONEV

Lecture 10

Duhamel's Principle

We will consider now solving inhomogeneous (forced) diffusion equations

$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = 0 \end{cases}$$

Note that if we want to solve with a non-zero initial condition we can just add to the above the solution of

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \varphi(x) \end{cases}$$

by the superposition principle

So all we need to do is nonzero f.

Before we tackle the PDE, let's do the same for ODEs

$$\begin{cases} y'(t) + a\,y = F(t), & t>0 \\ y(0) = 0 \end{cases}$$

Integrating factor is e^{at}

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}F(t)$$

$$\Rightarrow \boxed{y(t) = \int_0^t e^{-a(t-\tau)}F(\tau)\,d\tau}$$

Compare this to the solution of the IVP:

$$\begin{cases} w' + a\,w = 0 \\ w(0) = w_0 \end{cases}$$

$$w(t) = w_0\, e^{-at}$$

If we write this as

$$w(t;\tau) = F(\tau)\, e^{-at}$$

then

$$y(t) = \int_0^t w(t-\tau;\tau)\,F(\tau)\,d\tau$$

This means that if we now the solution of the homogeneous equation, we can use that to obtain the solution of the inhomogeneous equation

For ODEs the process of variation of constants is getting a particular solution of inhomog. eq. from the homog. eq.

PDEs can be seen as an infinite dimensional system of ODEs, so we expect something similar will apply.

Let us generalize this to PDEs (linear!)

Let us denote with

$$\mathcal{F}_t$$

the solution <u>operator</u> for the homogeneous equation.

that is $\qquad$ forward map

$$\boxed{\mathcal{F}_t(\varphi(\cdot))(x) = u(x,t)}$$

if

$$\begin{cases} u_t = k u_{xx} & \mathcal{L}u = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

Observe that $\mathcal{F}$ is a <u>linear operator</u>

We have an explicit formula

$$\boxed{\mathcal{F}_t \, \varphi(x) = \int_{-\infty}^{\infty} G(x-y,t)\, \varphi(y)\, dy}$$

But this is more general

Claim

The solution of

$$\begin{cases} \mathcal{L}u = f(x,t) \\ u(x,0) = \varphi(x) \end{cases}$$

is

$$\boxed{u(x,t) = \mathcal{F}_t(\varphi) + \int_0^t \mathcal{F}_{t-s} f(x,s)\,ds}$$

This is Duhamel's formula and is an important result that is quite general

For the heat equation

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\varphi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)\,f(y,s)\,dy\,ds}$$

This tells us that the Green's function $G(x,t)$ contains <u>all</u> the information necessary to solve parabolic linear PDEs. We will give a physical explanation soon. 6

To prove what we did was right we need to prove that

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y, t-s) f(y,s)\, dy\, ds$$

solves the PDE <u>and</u> satisfies $u(x,0) = 0$

This last piece is trivial, since for $t=0$ we have an integral from 0 to 0.

The harder piece is showing that

$$u_t + k u_{xx} = f(x,t)$$

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The tricky part here is that

$$\frac{\partial u}{\partial t} = \int_0^t \int_{-\infty}^{\infty} \frac{\partial G}{\partial t}(x-y, t-s) f(y,s)\,dy\,ds$$

$$\boxed{\frac{\partial G}{\partial t} = k \frac{\partial^2 G}{\partial x^2}}$$

$$\lim_{s \to t} \int_{-\infty}^{\infty} G(x-y, t-s) f(y,s)\,dy$$

The first part is just $k \dfrac{\partial^2 u}{\partial x^2}$ and this is not surprising since G solved $u_t = k u_{xx}$ and here we have an integral of G so by the superposition principle it will satisfy the heat equation also.

The inhomogeneous part comes from

$$f(x,t) = \int_{-\infty}^{\infty} G(x-y,0)\, f(y,t)\, dy$$

which we will understand next lecture

<u>hint:</u> $$f(x) = \int \delta(x-y)\, f(y)\, dy$$

The "variation of constants" formula is general, so let's use it for the

<u>wave equation</u>

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

Let's first rewrite as a system of first-order-in-time:

$$\begin{cases} u_t = v \\ v_t = c^2 u_{xx} + f(x,t) \end{cases}$$

$$\begin{cases} u(x,0) = 0 \\ v(x,0) = 0 \end{cases}$$

In order to solve this, we need to solve the homogeneous problem

$$\begin{cases} u_t - v = 0 \\ v_t - c^2 u_{xx} = 0 \\ u(x,0) = 0 \quad \text{(since no forcing)} \\ v(x,0) = f(x,\tau) \end{cases}$$

i.e. we need to solve

$$\begin{cases} w_{tt} = c^2 w_{xx} \\ w(x,0) = 0 \\ w_t(x,0) = \varphi(x) \end{cases}$$

Which we know how to solve already

$$w(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \varphi(s) \, ds$$

$$\mathcal{F}_t(\varphi)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(s)\,ds$$

Duhamel's formula says that

$$u(x,t) = \int_0^t \mathcal{F}_{t-\tau} f(x,\tau)\,d\tau$$

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\cdot d\tau$$

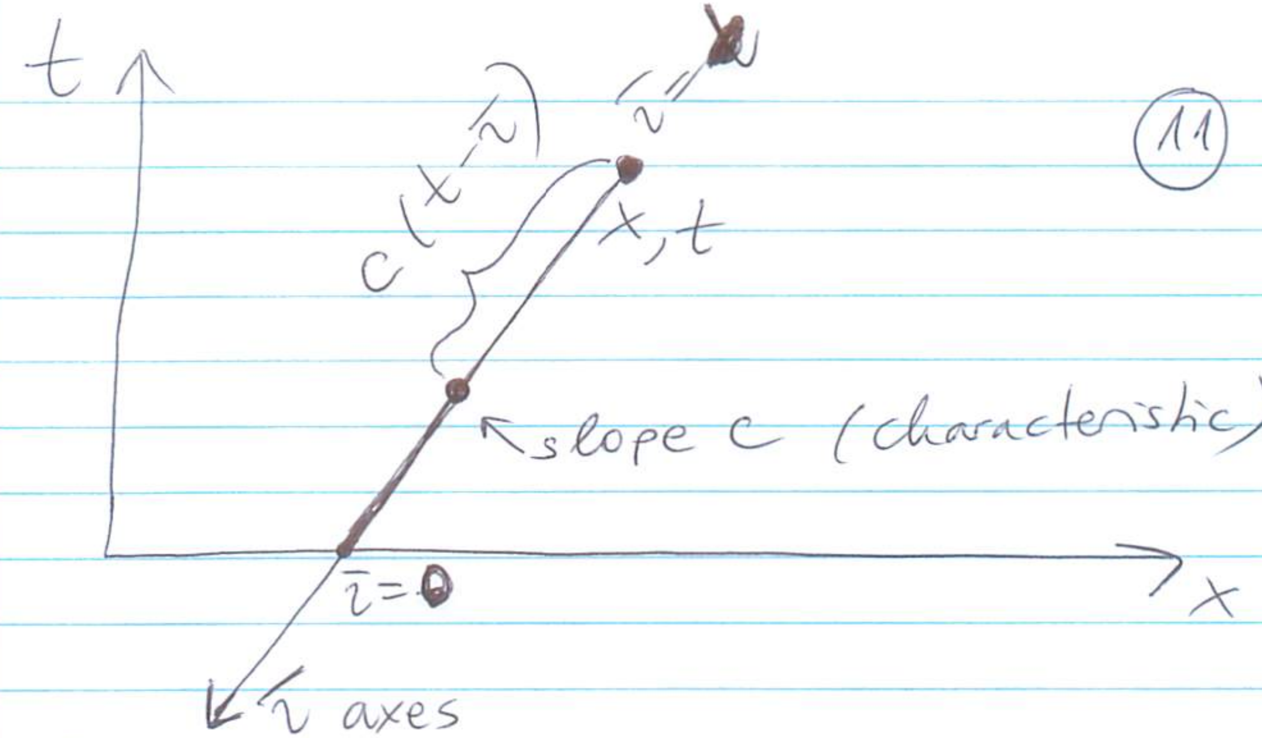
In homework you will repeat this for an advection equation to solve

$$u_t + c\,u_x = f(x,t)$$

to get

$$u(x,t) = \int_0^t f(x-c(t-\tau),\tau)\,d\tau$$

Which has a very <u>clear</u> physical interpretation



f is a source term - remember conservation law derivation.

The amount of "mass" injected during the time interval $d\bar\tau$ at time $\bar\tau$ at position x' is

$$f(x', \bar\tau)\, d\bar\tau$$

But the x' that will reach the point (x, t) is $x' = x - c(t - \bar\tau)$ which gives

$$f(x - c(t - \bar\tau), \bar\tau)\, d\bar\tau$$

Because of <u>linearity</u> we just add these up (integrate)