Why are Partial Differential Equations (PDEs) important?

They are the cornerstone of macroscopic (large/human scale) physics!

Examples of PDEs in action:

1. Weather prediction or climate modeling (Navier-Stokes equations)
2. Essentially all engineering, especially mechanical & aerospace elasticity (bridges) fluid flow (airplanes, rockets, ships)
3. Electromagnetism (Maxwell’s equations) (cell phones, radio communication)

Everything around us that we can see & internet with can be described by a PDE.
There are different kinds of PDEs related to different physics, e.g.,

1. Diffusion (heat conduction, pollutant dispersal, viscosity of liquids)
2. Advection (hurricanes, wind dispersal/flow, airplane design, traffic flow)
3. Deformation & Elasticity (electric fields, elastic materials, magnetism)

and others.

Course webpage:

http://cims.nyu.edu/~doney/Teaching/PDE

Visit the webpage ASAP

Three textbooks (PDE, Applied PDE, Essential PDE)
Ordinary Differential Equation (ODE):

Explicit: \[ \frac{du(t)}{dt} = f(t, u(t)) \]

Implicit: \[ F(t, u(t), \frac{du}{dt}, \frac{d^2 u}{dt^2}, \ldots, t) = 0 \]

For PDEs there is more than one dependent variable:

\[ u(x, y, z, \ldots, t) \]

\( x, y, z \) space \( t \) time

Notation:

\[ \begin{align*}
\frac{\partial u}{\partial x} &= u_x = \partial_x u \\
\frac{\partial^2 u}{\partial t \partial x} &= u_{tx} = u_{xt} = \partial_x u_t = \partial^2_x u_t
\end{align*} \]

\( \partial_x \partial_t \) twice differentiable

\[ \begin{align*}
\frac{\partial^2 u}{\partial x^2} &= u_{xx} = \partial_x^2 u \\
\frac{\partial^2 u}{\partial x^2 \partial t} &= u_{xxt} = u_{xtt} = \partial_t u_{xx} = \partial_x u_{tt} = \partial_x^2 u_{tt}
\end{align*} \]
Order of PDE

The order of a PDE is the highest order of derivative that appears in the equation.

General PDE:

\[ F(u, x, y, t, \ldots) = 0 \]

First order → \( u_x, u_y, u_t, \ldots \)

Second order → \( u_{xx}, u_{xy}, \ldots, u_{tt}, u_{tx}, \ldots \)

Third order → \( u_{xxx}, u_{xxy}, \ldots \)

Examples:

\( u_t = u_{xx} \) is second order

What does it mean to solve a PDE?

Try \( u(x, t) = e^{-t} \sin(x) \)

a) \( u_t = -e^{-t} \sin x = -u \)

b) \( u_x = e^{-t} \cos x \Rightarrow u_{xx} = -e^{-t} \sin x = -u \)

c) \( u_t = u_{xx} \) as needed
So $u(x,t) = e^{-t} \sin(x)$ is a solution of the PDE.

Is it the only? How do we choose among many (physical systems choose one after all)?

All of these are topics of the field of PDEs:

Mathematical Analysis

Practice

1. Show that $u = \ln(r)$, where $r = \sqrt{x^2 + y^2}$ solves

$$\nabla^2 u = 0 = \partial_{xx} u + \partial_{yy} u$$

2. Show that $u = 1/r$ solves

$$\nabla^2 u (x,y,z) = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u = 0$$

These are the Laplace equation in two and three dimensions.
Examples of PDEs

First order:

1. Advection equation (linear)
   \[ U_t + cU_x = 0 \]
   \( c > 0 \) is a constant (speed of flow) wind, flow in ocean, traffic (simple)

2. Inviscid Burgers (non-linear)
   \[ U_t + UU_x = 0 \]
   (traffic flow)

Second order:

1. Laplace's equation (linear)
   \[ U_{xx} + U_{yy} = 0 \text{ in 2D (two dimensions)} \]
   \[ U_{xx} + U_{yy} + U_{tt} = 0 \]

Short hand notation for Laplacian
\[ \nabla^2 u = \Delta u = 0 \]
Second order contd.

2. **Poisson's Equation** (e.g. electrostatic) (variant of Laplace)
\[ \nabla^2 u = \pm (x, y, z, t) \text{ LINEAR} \]

2*. **Non-linear Poisson:**
\[ \nabla^2 u = \pm (x, y, z, u) \text{ NON-LINEAR} \]

3. **Heat or diffusion eq.** (LINEAR)
\[ u_t = k u_{xx} \quad k > 0 \text{ is constant} \]

\[ \text{or generally} \quad u_t = k \nabla^2 u = k \Delta u \]

4. **Wave equation** (LINEAR)
(e.g. sound or **electrostatic waves**)
\[ u_{tt} = c^2 u_{xx} \]
\[ c = \text{sound speed} \]
\[ \text{or more general} \quad u_{tt} = c^2 \nabla^2 u \]
5. Advection–diffusion–reaction
    (pollutant transport) (linear)

\[ u_t + c u_x = k u_{xx} + f(u) \]

\[ \begin{align*}
    k &> 0 \quad \text{(diffusion coefficient)} \\
    f &\quad \text{(reaction rate)} \\
    c &\quad \text{(speed of flow)}
\end{align*} \]

or \( r_n = f(n) \) for nonlinear

6. Black–Scholes eq. (linear)
   (stock prices)

\[ u_t + \frac{\sigma^2}{2} u_{xx} + r x u_x = r u \]

\[ \begin{align*}
    \sigma &\quad \text{volatility} \\
    r &\quad \text{return rate}
\end{align*} \]

Third order

7. Korteweg de Vries (KdV) eq.

\[ u_t + 6u u_x + u_{xxx} = 0 \]

describes solitons
(e.g. lossless communication)
Exercise:
Show that \( u = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \) solves the heat equation
\[ u_t = u_{xx} \]

---

**Dimensional Analysis** (Physics)

- \( t = \text{time} = \; \text{seconds} \)
- \( x = \text{space} = \; \text{meters} \)
- \( \theta \) has units of \( \text{[m]} \)
- \( \theta x \)
- \( \theta \) has units of \( \text{[s]} \)
- \( \theta t \) has units of \( \text{[m}^{-2}] \)
- \( \theta x \theta x \)
- \( u \) has some units \( \text{[?]} \)
  - e.g., \( \text{kg, K (temperature)} \)
  - \( \text{mass} \)
\( u_t = ku_{xx} \)
\[
\begin{align*}
u_t &= \left[ \frac{u}{s} \right] = ku_{xx} = \left[ \frac{m}{s} \right] \left[ \frac{m^2}{s^2} \right] \\
\Rightarrow [k] &= \left[ \frac{m^2}{s} \right] \text{ (diffusion coefficient)}
\end{align*}
\]

Practice

What units does \( c \) have in

\[ u_t + cu_{xx} = 0 \]

What about

\[ u_t = c^2 u_{xx} \]

Principle (physics)

The answer must not depend on the choice of the units!

This means that special functions must have dimensionless arguments (no unit)

\[ u = \frac{1}{\sqrt{t}} e^{-x^2/4t} \] is not good

\[ \left[ \frac{x^2}{t} \right] = \left[ \frac{m^2}{s} \right] \]
But try

\[ u(x,t) = \frac{u_0}{\sqrt{kt}} e^{-x^2/(4kt)} \]

and show it solves

\[ u_t = ku_{xx} \]

New

\[ \sqrt{\left[ \frac{x^2}{kt} \right]} = \left[ \frac{m^2}{s} \right] = \text{no units (dimensionless)} \]

\[ \left[ \frac{1}{\sqrt{kt}} \right] = \left[ \frac{1}{\sqrt{\frac{m^2}{s}}} \right] = \left[ \frac{1}{m} \right] \]

So

\[ \left[ \frac{u_0}{\sqrt{kt}} \right] = \left[ \frac{u}{m} \right] \neq \left[ u \right] \]

so still doesn't work in terms of units.

We will see later in the class that the "right" solution is
\[
V(x, t) = \frac{U_0}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} ds
\]

\[
\begin{bmatrix}
\int ds
\end{bmatrix} = \begin{bmatrix}
ds
\end{bmatrix} = \begin{bmatrix}
m
\end{bmatrix}
\]

So \[
\begin{bmatrix}
U_0 \int ds
\end{bmatrix} = \begin{bmatrix}
m
\end{bmatrix}
\]

But is this a solution of PDE?

\[
V_t = -\frac{U_0}{2t \sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} ds
\]

\[
+ \frac{U_0}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{(x-s)^2}{4kt} e^{-\frac{(x-s)^2}{4kt}} ds
\]

\[
= \frac{U_0}{\sqrt{4\pi kt}} \left[ -\frac{1}{2t} \left( \begin{bmatrix}
1
\end{bmatrix} + \frac{1}{4kt^2} \left( \int (x-s)^2 ds \right) \right) \right]
\]

Practice: Compute \[V_{xx} = V_t\]