# PDE SPRING 2018

## LECTURE 1

Why are Partial Differential Equations (PDEs) important?

They are the cornerstone of macroscopic (large/human scale) physics!

Examples of PDEs in action:

1. Weather prediction or climate modeling (Navier-Stokes equations)
2. Essentially all engineering, especially mechanical & aerospace

   elasticity (bridges) — fluid flow (airplanes, rockets, ships)
3. Electromagnetism (Maxwell's equations) (cell phones, radio communication)

Everything around us that we can see & interact with can be described by a PDE

(2) There are different kinds of PDEs related to different physics, e.g.

(1) Diffusion
(heat conduction, polutant dispersal, viscosity of liquids

(2) Advection
(hurricanes, wind dispersal / flow, airplane design, traffic flow)

(3) Deformation & Elasticity
(electric fields, elastic materials, magnetism)

and others.

COURSE WEBPAGE:

http://CIMS.NYU.EDU/~DONEV/Teaching/PDE

$\boxed{\text{VISIT THE WEBPAGE ASAP}}$

Three textbooks (PDE, Applied PDE, Essential PDE)

③

# Ordinary Differential Equation (ODE):

Explicit: $$\frac{du(t)}{dt} = f\left(t, u(t)\right)$$

Implicit: $$F\left(t, u(t), \frac{du}{dt}, \frac{d^2u}{dt^2}, \ldots, t\right) = 0$$

For PDEs there is <u>more than one</u> dependent variable:

$$u(\underbrace{x, y, z}_{\text{space}}, \ldots, \underbrace{t}_{\text{time}})$$

<u>Notation</u>:

$$\frac{\partial u}{\partial x} \equiv u_x \equiv \partial_x u$$

$$\frac{\partial^2 u}{\partial t \, \partial x} \equiv u_{tx} = u_{xt} \equiv \frac{\partial^2 u}{\partial x \, \partial t}$$

twice differentiable

$$\frac{\partial^2 u}{\partial x^2} \equiv u_{xx} \equiv \partial_x^2 u$$

4

## Order of PDE

The order of a PDE is the highest order of derivative that appears in equation.

General PDE:

$$F(u, x, y, z, t, \ldots$$

first order $\rightarrow u_x, u_y, u_z, u_t, \ldots,$

second $\rightarrow u_{xx}, u_{xy}, \ldots, u_{tt}, u_{tx}, \ldots,$

third $\rightarrow u_{xxx}, u_{xyz}, \ldots) = 0$

Examples:

$u_t = u_{xx}$ is second order

What does it mean to solve a PDE?

Try $u(x,t) = e^{-t} \sin(x)$

(a) $u_t = -e^{-t} \sin x = -u$

(b) $u_x = e^{-t} \cos x \Rightarrow$

$u_{xx} = -e^{-t} \sin x = -u$

(c) $u_t = u_{xx}$ as needed

5

So $u(x,t) = e^{-t} \sin(x)$
is a solution of the PDE.

Is it the only? How do we choose among many (physical systems choose one after all)?

All of these are topics of the field of PDEs:
Mathematical Analysis

Practice

1. Show that $u = \ln(r)$, where $r = \sqrt{x^2+y^2}$, solves
$$\nabla^2 u = 0 = \partial_{xx} u + \partial_{yy} u$$

2. Show that $u = 1/r$ solves
$$\nabla^2 u(x,y,z) = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u = 0$$

these are the Laplace equation in two and three dimensions

6 Examples of PDEs

(A) First Order:

(1) Advection equation – (LINEAR)

$\boxed{u_t + c\,u_x = 0}$, $c > 0$ is a constant (speed of flow)

wind, flow in ocean, traffic (simple)

(2) Inviscid Burgers – (NON-LINEAR)

$u_t + u\,u_x = 0$

(traffic flow)

Second order

(1) Laplace's equation (LINEAR)

$u_{xx} + u_{yy} = 0$ in 2D (two dimensions)

$u_{xx} + u_{yy} + u_{zz} = 0$

Short hand notation for Laplacian

$$\boxed{\nabla^2 u = \Delta u = 0}$$

7

Second order contd.

2 Poisson's Equation (e.g. electrostatics)
(variant of Laplace)

$$\nabla^2 u = f(x, y, z, \ldots) \quad \text{LINEAR}$$

2* Non-linear Poisson:

$$\nabla^2 u = f(x, y, z, u) \quad \text{NON-LINEAR}$$

3 Heat or diffusion eq (LINEAR)

$$u_t = k\, u_{xx}, \quad k > 0 \text{ is constant}$$

or generally $\boxed{u_t = k \nabla^2 u \equiv k \Delta u}$

4 Wave equation (LINEAR)
(e.g. sound or electrostatic waves)

$$u_{tt} = c^2 u_{xx}, \quad c = \text{sound speed}$$
speed of light

or more general

$$\boxed{u_{tt} = c \nabla^2 u}$$

5 Advection - diffusion - reaction
(pollutant transport) (LINEAR)

$$u_t + c\,u_x = k\,u_{xx} + r\,u$$

$\begin{cases} k > 0 \\ r \\ c \end{cases}$ constants (diffusion coefficient) (reaction rate) (speed of flow)

Or $ru \to f(u)$ for nonlinear

6 Black - Scholes eq. (LINEAR)
(stock prices)

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + r x u_x = r u$$

option price volatility stock price return rate

## Third order

1 Korteveg de Vries (KdV) eq.

$$u_t + 6u\,u_x + u_{xxx} = 0$$

describes solitons
(e.g. lossless communication)

## Excercise:

Show that $u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$

solves the heat equation

$$u_t = u_{xx}$$

## Dimensional Analysis
(PHYSICS)

$t$ = time $= [s] = [\text{seconds}]$
units

$x \equiv$ space $= [m] = [\text{meters}]$

$\frac{\partial}{\partial x}$ has units of $[m^{-1}]$

$\frac{\partial}{\partial t}$ has units of $[s^{-1}]$

$\frac{\partial}{\partial x \partial x}$ has units of $[m^{-2}]$

$u$ has <u>some</u> units $[u]$

e.g. kg, K (temperature)
mass

10

$$u_t = k u_{xx}$$

$$u_t = \left[\frac{u}{s}\right] = k u_{xx} = [k]\left[\frac{u}{m^2}\right]$$

$$\Rightarrow \quad [k] = \left[\frac{m^2}{s}\right] \quad \left(\begin{array}{c}\text{diffusion}\\ \text{coefficient}\end{array}\right)$$

Practice

What units does $c$ have in

$$u_t + c u_x = 0$$

What about

$$u_t = c^2 u_{xx}$$

Principle (physics)

The answer must not depend on the choice of the units!

This means that special functions must have dimensionless arguments (no unit)

$$u = \frac{1}{\sqrt{t}} e^{-x^2/4t} \quad \text{is } \underline{\text{NOT}} \text{ good}$$

$$\left[\frac{x^2}{t}\right] = \left[\frac{m^2}{s}\right]$$

(11)

But try

$$u(x,t) = \frac{u_0}{\sqrt{kt}} e^{-x^2/(4kt)}$$

and show it solves

$$u_t = k u_{xx}$$

Now

$$\left[\frac{x^2}{kt}\right] = \left[\frac{m^2}{\frac{\cancel{m^2}}{\cancel{s}} \cdot \cancel{s}}\right] = \text{no units (dimensionless)} \quad \checkmark$$

$$\left[\frac{1}{\sqrt{kt}}\right] = \left[\frac{1}{\sqrt{\frac{m^2}{s} \cdot s}}\right] = \left[\frac{1}{m}\right]$$

So

$$\left[\frac{u_0}{\sqrt{kt}}\right] = \left[\frac{u}{m}\right] \neq [u]$$

so still doesn't work in terms of units

We will see later in the class that the "right" solution is

(12)

$$u(x,t) = \frac{u_0}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\, ds$$

$$\left[\int ds\right] = [ds] = [m]$$

So $\left[\frac{u_0 \int ds}{\sqrt{kt}}\right] = [u]$ ✓

But is this a solution of PDE?

$$u_t = -\frac{u_0}{2t\sqrt{4\pi k t}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\, ds$$

$$+ \frac{u_0}{\sqrt{4\pi k t}} \int \frac{(x-s)^2}{4kt} \underbrace{e^{-\frac{(x-s)^2}{4kt}}\, ds}_{\equiv \square}$$

$$= \frac{u_0}{\sqrt{4\pi k t}} \left[ -\frac{1}{2t}\left(\int \square\right) + \frac{1}{4kt^2}\left(\int (x-s)^2 \square\right)\right]$$

PRACTICE: Compute $u_{xx} \overset{?}{=} u_t$

Challenge