Partial Differential Equations, Spring 2020
Homework V: Separation of Variables

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Total number of points is 75.

1. (5 pts) Prove by direct integration that the complex exponential functions

$$u_m(x) = \frac{1}{\sqrt{2\pi}} \exp(i\, mx)$$

are orthonormal in L_2 on the interval $x \in [-\pi, \pi]$ for any integer $m \in \mathbb{Z}$. [Hint: *Remember that for complex valued functions the L_2 inner product involves a complex conjugate of one of the functions.*]

2. (10 pts) Find the complex (or periodic) exponential Fourier series of the function $\exp(-x)$ on the interval $[-1, 1]$, and then also express in terms of sines and cosines (this is called the "full Fourier" series). *Hint: See last page of Lecture 14 notes and next question.*

3. (7.5 pts) Show that a complex-valued function $f(x)$ on an interval $-L < x < L$ is real-valued *if and only if* the Fourier coefficients c_n in its exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(in\pi \frac{x}{L}\right)$$

satisfy the conjugacy condition $\overline{c_{-n}} = c_n$. It is OK to take $L = 1$ if you wish. [Hint: *Prove both directions of iff. For one of the two directions, try grouping terms with n and $-n$ together.*]

4. (12.5 pts) Solve the heat equation for a thin rod that is insulated at the ends:

$$u_t = u_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u_x(0, t) = u_x(L, t) = 0$$
$$u(x, 0) = 2 + \cos\left(\frac{5\pi x}{L}\right).$$

5. (27.5 pts) In this problem we consider the heat equation with several combinations of boundary conditions. [Hint: *Solving this does not require doing any integrals.*]

   (a) (15pts) Solve the diffusion equation with mixed Neumann-Dirichlet BCs:

$$u_t = 2u_{xx}, \quad 0 < x < \frac{1}{2}, \quad t > 0$$
$$u(0, t) = 0$$
$$u_x\left(\frac{1}{2}, t\right) = 0$$
$$u(x, 0) = \begin{cases} -1 & \text{for } 0 < x < \frac{1}{4} \\ 1 & \text{for } \frac{1}{4} < x < \frac{1}{2} \end{cases}.$$

(b) (5 pts) Explain (i.e., show that), without redoing any computations, why the same function $u(x,t)$ found in part 5a also solves this equation with Dirichlet BCs only:

$$\begin{aligned} u_t &= 2u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= \begin{cases} 1 & \text{for } \frac{1}{4} < x < \frac{3}{4} \\ -1 & \text{otherwise} \end{cases}. \end{aligned}$$

[Hint: *See section 3.1 in Strauss on using reflections to solve equations on a half line and apply to this problem.*]

(c) (7.5pts) Now solve the same equation as in part 5b with the same initial condition but this time with an inhomogeneous Dirichlet BC:

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 1. \end{aligned}$$

6. (12.5 pts) Find the eigenvalues of the first-derivative operator $\mathcal{L} = -\partial_x$ on the interval $(-1,1)$ with the boundary condition $X(-1) = X(1)$ (this is called periodic boundary conditions). Are the eigenfunctions orthogonal to each other? [Note: *This will be useful to us when we solve advection-diffusion type equations using computers. As a prelude, consider an operator that is a combination of a first and second or even third derivative, like $\mathcal{L} = a\partial_x + b\partial_{xx} + c\partial_{xxx}$ with periodic conditions — can you guess what the eigenfunctions are based on this problem and what we did for the Laplacian in class? Compute the eigenvalue also.*]