Total number of points is 65.

1. (7.5pts) Solve the wave equation $u_{tt} = c^2 u_{xx}$ on the whole real line, with the initial condition $u(x, 0) = e^{-2|x|}$ and $u_t(x, 0) = \sin x$.

2. (12.5 pts) Use a change of variables/coordinates $(x, y) \leftrightarrow (s, t)$ to determine the general solution (7.5pts) to the PDE

$$u_{xx} - 4u_{yy} = x - 2y. \quad (1)$$

Can you find a specific solution (5pts) in the whole $xy$ plane that satisfies the boundary conditions $u(x, 0) = x^2$ and $u(0, y) = y^2$?

[Hint: Recall it is a good idea to check the solution at the end.]

3. (10pts) Solve the diffusion equation $u_t = u_{xx}$ on the whole real line, with the initial condition

$$u(x, 0) = \exp (-|x|) \quad (2)$$

4. (10pts) Solve the PDE on the whole real line ($x \in \mathbb{R}, t > 0$)

$$u_t - u_{xx} = \begin{cases} 
\exp (-x) & \text{for } x > 0 \\
0 & \text{for } x \leq 0.
\end{cases}$$

with the initial condition (2).

5. (10pts) Use Duhamel’s principle to solve the PDE (1) with the boundary conditions

$$u(x = 0, y) = 0$$

$$u_x(x = 0, y) = 0.$$

[Hint: The formula for the answer is in the lecture notes on Duhamel’s principle.]

6. (15pts) Use Duhamel’s principle to solve the forced advection equation on the whole real line,

$$u_t + c u_x = f(x, t),$$

with an initial condition $u(x, 0) = 0$. Write down all steps of the derivation (10pts).

(5pts): Follow the method of characteristics to solve the PDE (and compare the answers, of course).