Partial Differential Equations, Spring 2020
Homework III: First-Order PDEs and Classification

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Due by Tuesday **March 3rd**, 2020

Total number of points is 60.

1. (7.5pts) Solve the PDE for $u(x,y)$
$$u_y + e^{x+2y} u_x = 0$$
over the whole (x,y) plane, with the boundary condition $u\,(x, y=0) = e^x$.
2. (7.5pts) Solve the PDE for $u(x,t)$
$$u_t + 2u_x - 2u = e^{t+2x}$$
over the whole (x,t) plane, with the initial condition $u(x,0) = 0$.
3. (5pts) Consider the advection-reaction (advection-decay) equation for $u(x,t)$
$$u_t + u_x = ru, \tag{1}$$
where r (reaction rate) is a constant. Introduce a new dependent variable $v(x,t) = u(x,t)\, e^{-rt}$, write the PDE in the new variable and solve it, and then solve the original PDE using this transformation. [Hint: *You can also solve it using the method of characteristics and compare.*]
4. (10pts) Consider the equation (1) in the positive quadrant of the space-time domain, $x > 0$ and $t > 0$, with the initial condition (IC)
$$u(x, t=0) = g(x) \quad \text{for} \quad x > 0$$
and the boundary condition (BC)
$$u(x=0, t) = h(t) \quad \text{for} \quad t > 0.$$
Draw the characteristics and use this with the IC and BC to write down the solution of the PDE.
5. (10pts) Among all second-order homogeneous PDEs in two dimensions with constant coefficients, show that the only ones that do not change under a rotation of the coordinate system (i.e., are *rotationally invariant*), have the form
$$\nabla^2 u = k\, u,$$
where k is some constant. What is the type (hyperbolic, parabolic, elliptic) of this PDE?
6. (10pts) Consider the PDE
$$u_{xx} - 2k u_{xy} + k^2 u_{yy} = 0,$$
where $k \neq 0$. What type is this PDE? Find a linear coordinate transformation $(x,y) \leftrightarrow (\eta, \theta)$ such that in the new coordinates the PDE is $u_{\eta\eta} = 0$. Use this to write the general solution of the PDE in the original coordinates.
7. (10pts) Consider the second-order PDE
$$u_t + \frac{1}{3} u_{xt} = 0.$$
   (a) [2pts] What is the type of this PDE?
   (b) [4pts] Solve it by introducing the variable $w = u_t$.
   (c) [4pts] Does a solution exist with the initial conditions $u(x,0) = e^{-3x}$ and $u_t\,(x,0) = 0$? If it does, is it unique?