Partial Differential Equations, Spring 2020
Homework II: Working with PDEs

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Due by Thursday **Feb. 20th**, 2018

Total number of points is 45.

1. (10pts) Show that the composition of divergence and gradient operators gives the Laplacian operator, $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla^2$, meaning that for any smooth scalar field we have $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u) = \nabla^2 u$ in two $(u(x, y))$ and three $(u(x, y, z))$ dimensions (5pts, 2.5pts each), and for any dimension d (5pts).
2. (7.5pts) Let us assume that the solution of the three-dimensional heat equation $v_t = k\nabla^2 v$ only depends on the distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$, i.e., $u(x, y, z, t) \equiv v(r, t)$. Perform the change of dependent variables to derive the PDE that $v(r, t)$ satisfies.
3. (7.5pts) Consider the two-point ODE boundary value problem (BVP) with one independent variable (i.e., really an ODE)
$$u''(x) + 2u = 0.$$
Find *all* solutions of this BVP with boundary conditions $u(0) = u(2) = 0$ (2.5pts) and also with BCs $u(0) = u(2\pi) = 0$ (5pts).
4. (5pts) What is the flux in the following conservation laws (2.5 per equation):
   (a) The viscous Burgers equation $u_t + 2u\, u_x = \kappa u_{xx}$.
   (b) The KdV equation $u_t + u\, u_x + \epsilon u_{xxx} = 0$.
5. (5 pts) Traffic can be modeled using the conservation law $u_t + \phi_x = 0$ for the density of cars u, where $\phi(u)$ is the flux function. The flux defines the average velocity of traffic $c(u)$ via $\phi = uc$. A reasonable model may be
$$c(u) = c_{\max}\left(1 - \left(\frac{u}{u_{\max}}\right)^2\right).$$
Derive the strong form of the conservation law (i.e., the PDE).
6. (10pts) Consider air flowing without any leaks/inflow through a straight pipe with variable thickness. If x denotes the position along the pipe, let $\rho(x, t)$ be the density (kilograms per volume, i.e., kilograms per meter cubed) of the air, and let $S(x)$ be the area of a cross-section of the pipe at position x. Let $f(\rho, x)$ denote the flow rate measured in mass per unit area per unit time (say kilograms per meter squared per second). Write down the flux function in terms of f and then show that ρ obeys the conservation law
$$\rho_t + f_x = -\frac{S'}{S} f.$$
List *all* steps. Check if the units make sense (this will help you get it right!).
7. (Challenge problem, won't be given points but recommended if you want to continue to PhD in math) Consider the Laplace equation $\boldsymbol{\nabla}^2 u = 0$ in a two-dimensional half space $y > 0$ (half plane above the x axes), with boundary conditions for both the value of the function and its y derivative:
$$u(x, y = 0) = f(x)$$
$$u_y(x, y = 0) = g(x).$$
Show that this is not a well-posed problem.
Hint: Consider solutions of the form (sinh denotes hyperbolic sine; confirm this satisfies PDE)
$$u(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \sinh(ny)$$
for very large n.