PDE A. DONEV

DUHAMEL'S PRINCIPLE

seen as superposition

We want to solve a linear inhomogeneous PDE but we already know how to solve the homogeneous one.

$$\begin{cases} u_t = \mathcal{L} u + f(x,t) \\ u(x, t=0) = \varphi(x) \end{cases}$$

We can use superposition to split into two pieces, one of which we know already:

$$u = v + w$$

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where

$$\begin{cases} v_t = \mathcal{L} v \\ v(x, t=0) = \varphi(x) \end{cases}$$

$\leftarrow$ homogeneous; we already know how to solve!

$$\begin{cases} w_t = \mathcal{L} w + f & \leftarrow \text{inhomogeneous PDE} \\ w(x, t=0) = 0 & \leftarrow \text{homogeneous IC} \end{cases}$$

So "without loss of generality" we can consider

$$\boxed{\begin{aligned} u_t &= \mathcal{L} u + f(x,t) \\ u(x, t=0) &= 0 \end{aligned}}$$

Imagine that we could solve:

$$(*)\begin{cases} u_t^{(\bar{\tau})} = \mathcal{L} u^{(\bar{\tau})} + \begin{cases} f(x,\bar{\tau}) & \text{if } \bar{\tau}-d\bar{\tau} \leq t < \bar{\tau} \\ 0 & \text{otherwise} \end{cases} \\ u(x,t=0) = 0 \end{cases}$$

where $\bar{\tau} > 0$ is a chosen time. This means the forcing $f(x,t)$ only acts very briefly over a time interval $d\bar{\tau}$ and then stops.

Note that for $t < \bar{\tau}$ we have $u^{(\bar{\tau})} = 0$ since there is nothing to force the solution away from zero

So we want to find

$$u^{(\tau)}(x, t \geq \tau)$$

Let us first compute

$$u^{(\tau)}(x, t = \tau)$$

because if we knew this then for $t \geq \tau$ we have the PDE

$$(**)\quad \begin{cases} u^{(\tau)}_t = \mathcal{L} u^{(\tau)}, & t \geq \tau \\ u^{(\tau)}(x, t = \tau) = \psi(x) \\ \quad = \text{given function} \end{cases}$$

which we already know how to solve

Over a very short time $d\overline{\tau}$ the term $\mathcal{L}u$ has no time to act, so approximately and with some hand-waving for $\overline{\tau}-d\overline{\tau}\leq t < \overline{\tau}$ we have

$$u_t^{(\overline{\tau})} \approx f(x,\overline{\tau}) = \text{constant in time}$$

$$\Rightarrow \quad u^{(\overline{\tau})}(x, t=\overline{\tau}) \approx f(x,\overline{\tau})\,d\tau$$

Going back to (**) we get (informally)

$$\begin{cases} u_t^{(\overline{\tau})} = \mathcal{L}u^{(\overline{\tau})}, & t > \overline{\tau} \\ u^{(\overline{\tau})}(x, t=\overline{\tau}) = f(x,\overline{\tau})\,d\tau \end{cases}$$

Add this later

We know how to solve this equation! (6)

Now, going back to (*)

we see that by superposition we can break $f(x,t)$ into short bursts of duration dt and then add these up.

This means that

$$\boxed{u(x,t) = \int_{\tau=-\infty}^{t} u^{(\tau)}(x,t)\, d\tau}$$

which is Duhamel's principle

For heat equation,

$$u^{(\tau)}(x, t>\tau) = \int_{-\infty}^{\infty} G(x-y, t-\tau) f(y,\tau)\, dy$$

but the principle is <u>general</u>