PDE Spring 2016

Lecture 7

The Wave Equation

We consider the wave equation in 1D in an \textit{unbounded} domain, i.e., the whole \textit{real} line.

Treating bounded domains is \textit{harder} and will be done later.

\[ u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad c \neq 0 \]

\( c \) = speed of wave (sound, etc.)

Statement

The general solution is of form:

\[ u(x,t) = f(x+ct) + g(x-ct) \]

for arbitrary functions \( f \) and \( g \) (continuously differentiable)

\[ = \text{LEFT WAVE} + \text{RIGHT WAVE} \]

\[ \leftarrow f(x+ct) \quad \text{and} \quad \rightarrow f(x-ct) \]
Let's check this is a solution:

\[ x = \partial^2_t - \partial^2_x = (\partial^2_t + c \partial^2_x)(\partial^2_t - c \partial^2_x) \]

\[ x = x^+ x^- \]

Observe

\[ x^+ g(x-ct) = \]

\[ (\partial^2_t + c \partial^2_x) g(x-ct) = 0 \]

since we know \( g(x-ct) \) solves the advection equation

\[ u_t + cu_x = 0 \]

Similarly,

\[ x^-(x+ct) = 0 \]

So

\[ x[t+g] = x^+ x^- [t+g] \]

\[ = x^+ x^- g(x-ct) = \]

\[ = x^+ (-cg(y) + g'(y)) = \]

\[ = x^+ \left( -h(x-c(t)) \right) = 0 \]
Now let us derive this solution and therefore show that it is general, i.e., every solution is of form

\[ u = f(x + ct) + g(x - ct) \]

Denote \( \mathcal{L}^+ u = \varphi = u_t + cu_x \)

\[ \Rightarrow \mathcal{L} u = \mathcal{L}^- \varphi = 0 \]

\[ \mathcal{L} u \iff \begin{cases} \mathcal{L}^+ u = \varphi \\ \mathcal{L}^- \varphi = 0 \end{cases} \]

i.e. we have converted the second-order PDE into a system of first-order advection equations, which we know how to solve.

This is the same as for ODEs

\[ u''(t) = f(u, t) \iff \begin{cases} u' = \varphi \\ \varphi' = f(u, t) \end{cases} \]
\[ \nabla^2 u = u_t - cu_x = 0 \Rightarrow \]
\[ u = h(x + ct) \Rightarrow \]
\[ u_t + cu_x = h(x + ct) \]

Try to show that this implies
\[ u = f(x + ct) + g(x - ct) \]

But in fact there is a simpler and more general way here:

\[ \begin{align*}
\text{Use characteristic coordinates} \\
\text{to transform PDE into the canonical form of hyperbolic PDEs} \\
U_{\xi \eta} = 0 \\
\end{align*} \]

\[ \xi, \eta \] eta
\[
\begin{align*}
    \xi &= x + ct \\
    \eta &= x - ct \\
    \partial_x &= \partial_\xi + \partial_\eta \\
    \partial_t &= c \partial_\xi + c \partial_\eta \\
    \Rightarrow & \quad \begin{cases} 
    \partial_x = -2c \partial_\eta \\
    \partial_t = 2c \partial_\xi 
    \end{cases} \\
    \partial_\eta u = -4c^2 \partial_\xi \partial_\eta u = 0
\end{align*}
\]

\[\Rightarrow \quad \partial_\eta \xi u = 0 = \partial_\xi (\partial_\eta u)\]

This is the first sort of equation we considered in this class, and we know this:

\[\partial_\eta u = v \Rightarrow \partial_\xi v = 0 \Rightarrow \]

\[v = f(\eta) = 2\eta u \Rightarrow \]

\[u = F(\eta) + G(\xi), \quad F' = f\]
\[ u = f(\eta) + g(\xi) \]
is the general solution, as we claimed.

\[ \eta = \text{const} \]
\[ x + ct = x_0 \]
\[ x - ct = x_0 \]

Information propagates at a maximum speed of \( c \)
(key property of hyperbolic PDEs)
We know $u = f(x+ct) + g(x-ct)$ but what are $f$ and $g$?

Since we have two unknown functions we need two ICs (since there are no boundaries we cannot specify a BC here)

\[
\begin{align*}
\begin{cases}
  u(x,0) &= \phi(x) \\
  u_t(x,0) &= \psi(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  u(x,0) &= f(x) + g(x) = \phi(x) \\
  u_t(x,0) &= c f'(x) - c g'(x) = \psi(x)
\end{cases}
\end{align*}
\]

Differentiate first equation to get the system

\[
\begin{align*}
\begin{cases}
  f' + g' &= \phi' \\
  f' - g' &= \psi / c
\end{cases}
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases}
  f' = \frac{1}{2} (\phi' + \psi / c) \\
  g' = \frac{1}{2} (\phi' - \psi / c)
\end{cases}
\]
\[ f = \frac{1}{2} \int - (y' + \frac{y}{c}) \, dx \]

\[ = \frac{1}{2} y(x) + \frac{1}{2c} \int_{0}^{x} y(s) \, ds + A \]

Similarly

\[ g = \frac{1}{2} y(x) - \frac{1}{2c} \int_{0}^{x} y(s) \, ds + B \]

\[ \Rightarrow f + g = y + (A + B) \Rightarrow A + B = 0 \]

(we got an extra degree of freedom here because we differentiated \( f + g = y \) first)

\[ u = f(x + ct) + g(x - ct) \]

\[ u = \frac{1}{2} \left[ y(x + ct) + y(x - ct) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} y(s) \, ds \]

d'Alambert's formula
We have now shown existence and uniqueness of the IVP (Cauchy problem for the wave equation).

Observe that d'Alambert's formula conforms to the picture about the domain of dependence:

\[ u \text{ or } \overline{u} \text{ outside the domain do not affect the solution} \]

**Example:** If \( \overline{u} = 0 \) then the initial profile \( u = \overline{u}(x,0) \) splits in half into two pieces.
How about stability?
Is the Cauchy problem well-posed?

Yes

Let \( u_1 \) and \( u_2 \) be two solutions for different ICs.

\[ u = u_1 - u_2 \] also a solution.

\[
|u(x,t)| \leq \max |\Psi| + \frac{1}{2c} \cdot \max |\Psi| \cdot \frac{2c t}{\text{width of interval}}
\]

\[ \Rightarrow \]

\[ \max |u_1 - u_2| < \max |\Psi_1 - \Psi_2| + T \max |\Psi_1 - \Psi_2| \]

where \( 0 < t < T \)

This means that for finite time small perturbations of the ICs induce small perturbations in solution.