Conservation Laws

\[ \text{EPDE: 3.2 / 3.2.1} \]

Reading: \text{APDE: Section 1.2, Ex. 1.15, 1.3, 1.7}

We consider the transport of a conserved quantity in 1D.

Examples:

\[ \rightarrow \text{Traffic flow} \]

\[ \rightarrow \text{Pollutant / smoke in air flow} \]

\[ \rightarrow \text{Heat (energy) flow through metal} \]

\[ \text{Traffic lane on Highway} \]

\[ u(x,t) = \text{density of cars/molecules} \]

\[ \text{Quantity} = \int u(x,t) \, dx \]

\[ \text{Flux} \quad \Phi(x,t) = \text{amount of quantity passing through } X \]
Positive means to the right (2)

E.g.: Number of cars passing through intersection/mile marker per second

\[ f(x,t) = \text{source/sink of quantity} \]

E.g.: Traffic coming onto highway from entrance (+) or leaving from exit (-)

Fundamental conservation law

\[ \frac{d}{dt} \int_{a}^{b} u(x,t) \, dx = \int_{a}^{b} f(x,t) \, dx \]

This is called a "weak form" of a conservation PDE. Under some assumptions, it can be
converted into a traditional PDE, which is called the "strong form" and is the focus of this course.

Recall from ODEs that

\[ x'(t) = f(x, t) \]

is equivalent to the integral equation

\[ x(t) = x(0) + \int_0^t f(x(s), s) \, ds \]

which is in fact more general than ODE (e.g. stochastic ODEs).

If function is sufficiently smooth, i.e., sufficiently continuously differentiable, we can convert conservation law to a PDE.

Lots of PDEs in practice come from conservation laws!
\[
\frac{d}{dt} \int_a^b u(x,t) \, dx = \int_a^b u_t \, dx
\]

if \( u \) has continuous first partial derivatives.

Similarly, if \( \psi \) has continuous first partials, then from th. of calculus says:

\[
\psi(a,t) - \psi(b,t) = - \int_a^b \psi_x \, dx
\]

Conservation law becomes:

\[
\int_a^b [u_t + \psi_x - f] \, dx = 0
\]

for all \( a \) and \( b \)

Since integrand is continuous (crucial!) and \( a \) and \( b \) are arbitrary, the above implies integrand identically vanishes.
\[ \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \]

**Conservation Law PDE**

To turn this into a PDE we need a _constitutive law_

\[ f = f(u, x, t) \]

which is problem-specific.

For traffic flow

\[ f = uc \]

where \( c \) is speed of cars.

If \( c \) is constant (boring!) then we get an advection equation

\[ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \]

But in the real world \( c \) depends on \( u \) (and also \( x, t \) due to road/weather conditions).
So to get an actual PDE we need to do mathematical modeling.

E.g.

\[ c = c_{\text{max}} \left( 1 - \frac{u}{u_{\text{max}}} \right) \]

where \( u_{\text{max}} \) is the "jamming" density of cars.

\[ \Rightarrow \] (derive as practice)

\[ u_t + c_{\text{max}} \left( 1 - 2 \frac{u}{u_{\text{max}}} \right) u_x = 0 \]

Speed of traffic "waves" or speed of propagation of information through the "highway".

The above is a non-linear advection equation similar to Burgers' equation.
We will not study nonlinear advection in detail but see text books for advanced students.

Here, as practice, let's show that the implicit solution

\[
\begin{align*}
\mathbf{u} &= \mathbf{F}(x - c(n)t) \\
\text{solves} \\
\mathbf{u}_t + c(n)\mathbf{u}_x &= 0
\end{align*}
\]

\[
\mathbf{u}_t = \mathbf{F}'(y) \left[ -c(n) - c(n)\mathbf{u}_t + t \right]
\]

\[
\mathbf{u}_x = \mathbf{F}'(y) \left[ 1 - c'(n)\mathbf{u}_x t \right]
\]

\[
c\mathbf{u}_x = \mathbf{F}'(y) \left[ c - c'c\mathbf{u}_x t \right]
\]

\[
\frac{\mathbf{u}_t + c\mathbf{u}_x}{\mathbf{F}'(y)t} \left[ \mathbf{u}_t + c\mathbf{u}_x \right] = 0
\Rightarrow \quad \text{if } c'\mathbf{F}' \neq 0 \quad \mathbf{u}_t + c\mathbf{u}_x = 0
\]
Diffusion

as a conservation law

We all know that heat "flows" from hot to cold.

In general, the higher the gradient of temperature the larger the flow.

So it seems reasonable to postulate

\[ q = -k \frac{dU}{dx} \]

where \( k > 0 \) is a diffusion constant

\( \Rightarrow \) flux is "down the gradient"

Heat equation as a conservation law

\[ \frac{\partial U}{\partial t} + (kU_x)_x = 0 \]

Here \( k = k(x) \) or even \( k = k(U,x,t) > 0 \) works.
If \( \frac{\partial^2 u}{\partial x^2} > 0 \)
\[ \Rightarrow u_t > 0 \]

If \( \frac{\partial^2 u}{\partial x^2} < 0 \)

\[ \xrightarrow{\text{as time goes}} \]

Diffusion "flattens" or "smears" the solution

It makes it smoother with time

There is a deep connection between diffusion and randomness or random walks, see 1.4 in Applied PDE book if interested in details
Higher dimensions

$\Omega = \text{boundary of } \mathbb{R}^2$

$$\frac{1}{dt} \int \mathbf{u} \cdot d\mathbf{v} = \int (\mathbf{\Psi} \cdot \mathbf{n}) ds$$

where $V \subset \Omega$ is an arbitrary subvolume

Here flux $\mathbf{\Psi}$ is a vector and $\mathbf{n}$ is the normal flux through the boundary of a subvolume
Since $V$ is fixed with time,
\[ \frac{d}{dt} \int_V u \, dv = \int_V \frac{\partial u}{\partial t} \, dv \]

Also recall divergence theorem:
\[ \int_V \nabla \cdot \mathbf{v} \, dv = \int_{\partial V} \mathbf{v} \cdot n \, ds \]

\[ \int_V (\nabla \cdot \mathbf{v}) \, dv = \int_{\partial V} \mathbf{v} \cdot n \, ds \]

\[ \int_V \left( \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{v} - f \right) \, dv = 0 \]

for all $V \subseteq \mathbb{R}^n$ (weak form)

For certain assumptions:

\[ \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{v} = f \]

strong form
A direction in higher dimensions \( c \Rightarrow \vec{c} \) (vector)
\( \vec{c}(x,y) \Rightarrow \text{vector field in general} \)
\( \nabla \vec{c} = u \vec{c} \)

\[ \Rightarrow \quad u_t + \nabla \cdot (u \vec{c}) = 0 \]
if \( \vec{c} \) is a constant vector

\[ u + \vec{c} \cdot (\nabla u) = 0 \]

Diffusion in higher dimensions
\( \vec{V} = -k \nabla u , \quad k > 0 \)
\[ u_t + \nabla \cdot (-k \nabla u) = 0 \]
if \( k \) constant
\[ u_t = k \nabla \cdot (\nabla u) = k \nabla^2 u \]

Practice: Show \( \nabla \cdot \vec{V} = \nabla^2 \vec{V} \)
Higher dimensions (heat eq) 

\[ \nabla^2 u = 0 \]

Dirichlet boundary condition: 
\[ u(x, y) = \begin{cases} 
\text{fixed value} & \text{on } \partial \Omega_1 \\
\end{cases} \]

Neumann boundary condition (flux BC): 
\[ \frac{\partial u}{\partial n} = (\nabla u) \cdot n = g_2(r) \]

\[ g_2 = \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \]
\( \nabla_3: \text{ Robin } \Rightarrow \nabla_3 \)

\[ \frac{\partial^2 u}{\partial t^2} u(\tau) + \beta \frac{\partial^2 u}{\partial \tau^2} = g(\tau, t) \]

The Laplace equation \( \nabla^2 u = 0 \) is the steady state \( (t \to \infty) \)

- limit of the heat eq.
- no IC any more (no time)
- but the same BC's apply

Same goes for Poisson \( \nabla^2 u = f(\tau) \)

How many BCs do we need?

- depends on equation in non-trivial ways

\[ u_t + c u_x = 0, \quad c > 0 \]

\[ \text{IC} \]

\[ \text{BC} \]

\[ \text{No BC needed here for } c > 0 \]