Lecture 3  PDE Classification

Linear differential equations

Operator: infinite-dimensional generalization of functions, or "function of a function"

\[ L(u) = Xu = +kU_{xx} \]

Caligraphic letter

\( D, D^2, \frac{\partial^2}{\partial x^2} \) or \( \Delta \)

grad, \( \Delta \) or Laplacian

\[ u_t = Xu \leq \text{heat equation} \]

or \( Xu = \nabla^2 u_t - kU_{xx} \)

\[ L = \partial_t - k\Delta \]

\[ Xu = 0 \] Heat eq. m operator notation
Boundary condition

$$B \ u = \begin{cases} u(0, t) & \forall t > 0, x = 0 \\ u_x(1, t) & \forall t > 0, x = 1 \end{cases}$$

B.c. $$B \ u = f(x, t)$$

\[ f(0, t) = \begin{array}{c} f_1(t) \\ f(1, t) = f_R(t) \end{array} \]

$\mathcal{L}$ is a linear operator if:

\[ \begin{align*}
\mathcal{L}(x \ u) &= x \mathcal{L}(u) \\
\mathcal{L}(u + v) &= \mathcal{L}(u) + \mathcal{L}(v)
\end{align*} \]

Check $\mathcal{L} = \partial_{xx}$ follows from proper $\mathcal{L}$.

Check $\mathcal{L}(u) = uu_x$

\[ \mathcal{L}(x \ u) = (x \ u)(x \ u_x) = x^2 \mathcal{L}(u) \]

Not linear.
Standard Dirichlet, Neumann & Robin BCs are linear

A PDE is linear if it has the form

\[ \nabla u = f(x,t) \]

\[ \frac{\partial u}{\partial t} = h(x,t) \]

If \( f(x,t) = 0 \) then the PDE is homogeneous (otherwise non-homogeneous).

All boundary and initial data must be zero.
Superposition principle

1. If \( U_1 \) and \( U_2 \) are two solutions of a linear PDE, then so is any linear combination of them.
   \[
   U = \alpha U_1 + \beta U_2 \quad \text{since} \quad \Delta U = \Delta (\alpha U_1 + \beta U_2) = \alpha \Delta U_1 + \beta \Delta U_2 = 0
   \]

2. If \( U \) is a "particular" solution of \( \Delta U = f \) (non-homogeneous), then if \( \Delta U = 0 \) we have \( \Delta (U + U_\text{particular}) = f \) is another solution.

   \( \iff \) consequence

3. \( \Delta U = f \) has a unique solution \( U = 0 \) is the only solution of \( \Delta U = 0 \).

   \( \text{Proof:} \) Suppose \( U_1, U_2 \) are solns.
   \[
   \Delta U_1 = f, \quad \Delta U_2 = f \Rightarrow \Delta (U_1 - U_2) = 0 \Rightarrow U_1 = U_2
   \]
Linear BVPs have zero, one or infinitely many solutions (just like linear systems $A\mathbf{x} = \mathbf{b}$).

**Well-posedness**

A BVP which has a unique solution that varies continuously with the initial and boundary data is well-posed, otherwise ill-posed.

\[ Y u = f \]
\[ Y (u + \delta u) = f + \delta f \]

\( \delta f \) small $\Rightarrow$ \( \delta u \) small

If well-posed

\[ \| \delta u \| \leq C \| \delta f \| \]

\( u_t = u_{xx} \) — well-posed

But \( u_t = -u_{xx} \) (backward heat)

not well-posed

\( u(x,0) = \alpha \cos(nx) \Rightarrow u(x,t) = \alpha \varepsilon^{n^2 t} \cos(nx) \)

\( n \) large = large growth.
Continuum superposition

\[ u(x,t; \kappa_n) \text{ is a solution} \]

\[ u = \int \kappa \cdot u(x,t; \kappa_n) \, d\kappa \]

\[ L = \int L \cdot u(x,t; \kappa_n) \, d\kappa = 0 \]

If \( w = u + i v \) is a complex solution then \( u \) and \( v \) are real solutions.
Classification of PDEs

Linear vs. Non-linear
(also semi-linear & quasi-linear)

But then comes physics!

wave-like (hyperbolic)
diffusion-like (parabolic)
equilibrium (steady state) (elliptic)
Wave equation classification:

$$u(x,t) = A e^{i(kx - wt)}$$

is called a **plane wave** solution.

- $k = \text{wave number} = \frac{2\pi}{\lambda}$, wave length
- $w = \text{frequency} = \frac{2\pi}{T}$, wave period
- $\omega = 2\pi f$, frequency

$$u_t = D u_{xx}$$

$$-i \omega u = -D k^2 u \Rightarrow w = -i D k^2$$

$$w = w(h) \leq \text{dispersive relation}$$

If $w(h)$ is complex $\Rightarrow$ **diffusive**

$$u_t + u_{xxx} = 0 \Rightarrow +i \omega u = i D k^3 u$$

$$w = k^3$$

If $w(h)$ is real $\Rightarrow$ **wave-like**

$$w''(h) \neq 0 \Rightarrow \text{dispersive wave eq.}$$

**Practice:**

$$\begin{cases}
    u_t + cu_x = 0 \\
    u_{tt} = c^2 u_{xx}
\end{cases}$$
Change of variables

\[ u_t = u_x^2 + uu_{xx} \]

\[ w = e^u \]

**First way:**

\[ w_t = (e^u) u_t = w u_t = \]

\[ = w (u_x^2 + uu_{xx}) \]

\[ w_x = w u_x \]

\[ w_{xx} = w_x u_x + w u_{xx} = \]

\[ = w u_x^2 + w u_{xx} = w (u_x^2 + u_{xx}) \]

\[ \Rightarrow w_t = w_{xx} \rightarrow \text{heat eq.} \]

**Second way**

\[ n = \ln w \quad \text{where} \quad w > 0 \]

\[ n_t = \frac{1}{w} w_t \]

\[ u_x = \frac{1}{w} w_x \]

\[ u_{xx} = \frac{u_{xx}}{w} = \frac{1}{w^2} w_x^2 \]

\[ \frac{1}{w} w_t = \frac{4}{w^2} w_x^2 + \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2 \]

\[ \Rightarrow w_t = w_{xx} \quad \text{if} \quad w \neq 0 \]