Let's start from the Laplace equation

\[ \nabla^2 u = 0 \text{ in } \Omega \]

\[ u (\partial \Omega) = g (\partial \Omega) \]

Dirichlet BCs

In one-dimension, the Laplace equation is trivial:

\[ u'' = 0 \implies u = ax + b \text{ (linear)} \]
But in two and three dimensions it is much more interesting.

Functions \( u(x,y) \) that satisfy

\[
\nabla^2 u = U_{xx} + U_{yy} = 0
\]

are called **harmonic functions** and are central to complex analysis.

The Laplace equation is often solved by separation of **variables**.

**E.g.,**

Solve \( \nabla^2 u = 0 \) on rectangle

\( R = [0, a] \times [0, b] \)

**inhomogeneous** with **Dirichlet BCs on the sides of the rectangle**
We can split this problem into four simpler subproblems by the superposition principle.

So it is sufficient to consider non-homogeneous BCs on only one side of the rectangle.
\[ \nabla^2 u = 0 \]

\[ \begin{cases} 
   u(x, 0) = u(a, y) = u(0, y) = 0 \\
   u(x, 0) = g(x) \\
   \text{Look for separable solutions} \\
   u(x, y) = \overline{X}(x) \overline{Y}(y) \\
   \Rightarrow \begin{cases} 
   \overline{X}(0) = \overline{X}(b) = 0 \\
   \overline{Y}(b) = 0 \\
   \text{but } \overline{Y}(0) \text{ is undetermined} \\
   \end{cases} \\
   \overline{u}(x, 0) = \overline{X}(x) \overline{Y}(0) \\
   \text{set } |\overline{Y}(0)| = 1 \\
\end{cases} \]

\[ u_{xx} + u_{yy} = \overline{X}'' \overline{Y} + \overline{X} \overline{Y}'' = 0 \]

\[ \Rightarrow -\frac{\overline{X}''}{\overline{X}} = \frac{\overline{Y}''}{\overline{Y}} = \text{constant} = \lambda \]
We get the one-dimensional eigenvalue problem

\[ \begin{cases} x'' = -\lambda x \\ x(0) = x(a) = 0 \end{cases} \]

which we have already solved

\[ \begin{cases} X_n = \sin \left( \frac{n\pi x}{a} \right) \\ \lambda_n = \left( \frac{n\pi}{a} \right)^2 \end{cases} \]

\[ Y'' = \lambda_n Y(y) \]

positive sign

\[ Y = c_1 e^{\sqrt{\lambda_n}y} + c_2 e^{-\sqrt{\lambda_n}y} \]

\[ Y(b) = c_1 e^{\sqrt{\lambda_n}b} + c_2 e^{-\sqrt{\lambda_n}b} = 0 \]

\[ \Rightarrow c_1 = -c_2 e^{-2\sqrt{\lambda_n}b} \]

\[ Y = c_2 \left[ e^{\sqrt{\lambda_n}(y-2b)} - e^{-\sqrt{\lambda_n}y} \right] \]
Usually written as
\[ Y = C_2 e^{-\sqrt{\lambda n} b} \left[-e^{\sqrt{\lambda n} (y-b)} - e^{-\sqrt{\lambda n} (y-b)}\right] \]

Denoting
\[ \sinh x = \frac{e^x - e^{-x}}{2} \]

we can write
\[ Y = C \sinh \left( \sqrt{\lambda n} (y-b) \right) \]

From
\[ Y(0) = 1 = C \sinh \left( \sqrt{\lambda n} b \right) \]

\( C = -\frac{1}{\sinh \left( \sqrt{\lambda n} b \right)} \)

Final answer for eigenfunctions
\[ \begin{align*}
U_n(x,y) &= \frac{\sinh \left( n\pi (1-y/b) / a \right)}{\sinh \left( n\pi / a \right)} \sin \left( \frac{n\pi x}{a} \right) \\
n &= 1, 2, \ldots
\end{align*} \]
We hope we can expand the solution as a sum of these
\[ U = \sum_{n=1}^{\infty} A_n \, U_n(x,y) \]

\[ U(x,0) = g(x) \quad \text{BC} \]

\[ g(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi x}{a} \right) \]

\[ \Rightarrow \quad g(x) = \sum_{n=1}^{\infty} \left[ \frac{2}{a} \int_{0}^{a} g(x) \sin \left( \frac{n \pi x}{a} \right) dx \right] \sin \left( \frac{n \pi x}{a} \right) \]

\[ \frac{\sinh \left( \frac{n \pi (1-y/e)}{a} \right)}{\sinh \left( \frac{n \pi}{a} \right)} \]

is the solution of Laplace's equation
Now let's consider the __Poisson equation__ on a rectangle:

\[
\begin{align*}
\Delta u &= u_{xx} + u_{yy} = f(x,y) \\
\n\end{align*}
\]

\[
\begin{align*}
\n\end{align*}
\]

\[
\begin{align*}
\text{Idea: To solve } \Delta u &= f, \\
\text{first find the eigenfunctions of } \Delta, \text{ and then expand both solution and rhs in that basis.}
\end{align*}
\]

\[
\begin{align*}
u &= \sum_{n} a_{n} u_{n} \\
f &= \sum_{n} b_{n} u_{n}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \Delta u &= \sum_{n} a_{n} (\Delta u_{n}) = \\
&= \sum_{n} \lambda_{n} a_{n} u_{n} = f = \sum_{n} b_{n} u_{n}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \text{by orthogonality,} \\
b_{n} = \lambda_{n} a_{n} = \frac{b_{n}}{\lambda_{n}}
\end{align*}
\]
This assumes that $\lambda = 0$ is not an eigenvalue, which is a sufficient and necessary condition for $\lambda u = f$ to have a unique solution.

$$u = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} u_n$$

From orthogonality, from

$$f = \sum b_n u_n$$

$$b_n = \frac{(u_n, f)}{(u_n, u_n)}$$

$$(u_n, f)$$ is orthonormal basis

$$u = \sum_{n=1}^{\infty} \frac{(u_n, f)}{\lambda_n} u_n$$
The finite-dimensional linear algebra version of this is:

$$A \mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1} \mathbf{b}$$

$$A = U \Lambda U^*$$ if $A$ unitarily diagonalizable

$$\implies A^{-1} = U \Lambda^{-1} U^*$$

Since

$$A A^{-1} = U \Lambda U^* U \Lambda^{-1} U^* = U U^* = I$$

$$\implies x = A^{-1} b = U \Lambda^{-1} (U^* b)$$

We don't usually solve linear systems in $\mathbb{R}^n$ this way since there are faster ways, but for PDEs it is the best way to construct an "inverse Laplacian"

$$\mathcal{U} = \Delta^{-2} f \quad \text{(notation)}$$
The only missing piece is to construct the eigenfunctions of the Laplacian on a rectangle with homogeneous Dirichlet BCs.

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda \mu_n, \quad u(\partial R) = 0 \]

Take

\[ u = X(x)Y(y) \]

separable (only works because domain is so simple)

\[ \Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} - \mu = \lambda \]

\[ \Rightarrow \begin{cases} X'' = -\lambda X \Rightarrow \lambda = \left( \frac{m \pi}{a} \right)^2 \\ X(0) = X(a) = 0 \end{cases} \]

New

\[ \begin{cases} Y'' = (\mu - \lambda) Y \\ Y(0) = Y(b) = 0 \end{cases} \]

slightly shifted eigenvalue

\[ \Rightarrow \mu - \lambda = \left( \frac{m \pi}{b} \right)^2 \]
\[ U_{m,n} = \left( \frac{\pi n}{a} \right)^2 + \left( \frac{\pi m}{b} \right)^2 \]

\[ U_{m,n} = \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right) \]

\[ f = \sum b_{m,n} U_{m,n} \]

\[ b_{m,n} = \left( \frac{f, U_{m,n}}{U_{m,n}, U_{m,n}} \right) \]

\[ \iint f(x,y) \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right) \, dx \, dy \]

\[ b_{m,n} = \frac{1}{4 \, ab} \]

and \[ n = \sum \frac{b_{m,n}}{U_{m,n}} U_{m,n} \]

In this case the \( x \) and \( y \) direction completely separate.
Note that the Poisson equation with inhomogeneous BCs is, by superposition, a sum of Laplace with inhomogeneous BCs and Poisson with homog. conditions

\[ \begin{align*}
  \Delta^2 u &= f \\
  u(\partial \Omega) &= \psi(\partial \Omega) 
\end{align*} \]

\[ u = u_n + u_2 \]

\[ \begin{align*}
  \Delta^2 u_n &= f \quad \text{and} \\
  u_n(\partial \Omega) &= 0 \\
  \Delta^2 u_2 &= 0 \\
  u_2(\partial \Omega) &= \psi
\end{align*} \]

We will not prove it here but it is not hard to show that the Poisson \& Laplace equations with Dirichlet BCs are well-posed, notably the solution is unique (we already proved the maximum principle for Laplace's equation \& this can be used in a proof)
A side note: Green's Functions

Assume that we could find the Green's function for the Poisson equation with homogeneous BCs

\[ \nabla^2 G = \delta(x - x_0, y - y_0) \text{ point source} \]

\[ (x_0, y_0) \in \mathbb{R}^2 \]

\[ G(\partial \mathbb{R}^2) = 0 \quad (\text{homogeneous Dirichlet}) \]

Since

\[ f(x, y) = \iint f(x_0, y_0) \delta(x - x_0, y - y_0) \]

\[ (x_0, y_0) \in \mathbb{R}^2 \quad dx_0 \, dy_0 \]

by superposition \( \sum n = f \Rightarrow \)

\[ u(x, y) = \iint f(x_0, y_0) G(x, y; x_0, y_0) \, dx_0 \, dy_0 \]

\[ (x_0, y_0) \in \mathbb{R}^2 \]
We usually write this in vector notation

\[ G(x, y; x_0, y_0) = G(x, x_0) \]

Recall our previous formula

\[ u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \| u_n \|^2} \left[ \int f(y) u_n(y) \, dy \right] u_n(x) \]

\[ = \int \left( \sum_{n=1}^{\infty} \frac{u_n(y) u_n(x)}{\lambda_n \| u_n \|^2} \right) f(y) \, dy \]

Defines

\[ G(x^2, y^2) \]