Sturm-Liouville Problems

As motivation let us consider the heat equation on a circle with a radially-symmetric IC & BC.

E.g.

\[ u_t = u_{xx} + u_{yy} = \Delta u \]

for \( x^2 + y^2 < r^2 \)

\[ u(\partial \Omega) = 0 \]

\[ u(\Omega, t=0) = f(\Omega) \]

We know that the solution is spherically-symmetric so we should use polar coordinates.
\[ \begin{align*}
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta 
\end{cases}
\end{align*} \]

\[-\Delta u = - \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \tag{from Calculus 3} \]

For us \( u = u(r) \) only, so the eigenvalue problem we need to solve is:

\[ \Delta u = \lambda u \quad \text{where} \]

\[ (\Delta u)(r) = - \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \]

\[ = - \frac{1}{r} \left( r u' \right)' = \lambda u \]

So we get:

\[ \begin{cases}
\frac{1}{r} \left( r u' \right)' + \lambda u = 0 \\
u(0) - n(r) = 0
\end{cases} \]

which is a one-dimensional problem (same in 3D but different coefficients)
These types of problems appear often in practice and we consider them in this class.

We consider the two-point BVP (i.e. ODE!)

\[ \begin{align*}
2u &= -(p(x)u'(x))' + q(x)u \\
&= -pu'' - p'u' + qu = f(x)
\end{align*} \]

on \( a \leq x \leq b \), with either Dirichlet, Neumann, periodic, or mixed BCs.

There are two cases to be considered:

1. **Non-singular**: 
   \[ p(x) > 0, \quad q(x) > 0 \quad \forall a < x < b \]
   \[ \text{cont. differentiable continuous} \]

2. **Singular** (harder!)
   \[ p(a) = 0 \quad \text{or} \quad p(b) = 0 \]
Observe that a general BVP in one dimension that is linear and second-order takes the form

\[
\frac{dx}{p(x)} = u'' + a(x)u' + b(x)u - q(x)
\]

With a trick we can write this in the SL form:

\[
p(x)u'' - p(x)a(x)u' - p(x)b(x)u = p(x)q(x)
\]

From \((pu')' = pu'' + p'u'\) we get

\[
pu'' = (pu')' - p'u'
\]

\[
(pu')' - (pa + p')u' - pbu = pg
\]

versus the SL equation

\[
(pu')' + gu = f(x)
\]

So we want \(pa + p' = 0\)
\[ p' = -a(x)p \Rightarrow \]
\[ p(x) = \exp\left[-\int a(x)\,dx\right] \]
\[ q(x) = p(x)g(x) \]
\[ f(x) = -p(x)g(x) \]

Observe that as long as \( a(x) \) is integrable, \( p(x) \) exists and is positive, so the Sturm-Liouville problem is quite general!

The key observation that will allow us to solve SL problems is to find the SL eigenfunctions & eigenvalues

\[ Lu = \lambda u + \text{BCs} \]

and then expand the RHS \( f(x) \) or initial condition for IVPs into an orthogonal series.
Eigenfunctions

The key observation is that:

The Sturm-Liouville operator is an
self-adjoint operator

\[ (u, xv) = \int_0^L \left[ - \left( p(x) \frac{du}{dx} \right)' + q(x) u \right] dx \]

\[ = -\int_a^b u \left( p(x) \frac{du}{dx} \right)' dx + \int_a^b q(x) u v dx \]
integrate by parts

\[ = \left[ p u v \right]_a^b \]

\[ + \int_a^b \left( p u \frac{dv}{dx} + q u v \right) dx \]
integrate by parts again

\[ = \left[ p \left( u v' - u' v \right) \right]_a^b + (xu, v) \]
Therefore
\[ (u, Xv) - (Xu, v) = \left[ p(u' \overline{v} - \overline{u}v') \right]_a \]

this vanishes for many BCs.

So for a number of common BCs we have
\[ (u, Xv) = (Xu, v) \implies X^* = X \implies \text{self-adjoint} \]

From the fact \( X \) is a symmetric (self-adjoint) operator we already know a lot of consequences.

E.g. it's eigenvalues are real, eigenvectors are orthogonal (even complete), etc.

Turns out we knew their sign also.
Theorem:

If \( p(x) > 0 \), \( q(x) > 0 \) for \( x \in (a, b) \), then the eigenvalues are real and positive.

This means the operator is symmetric positive definite.

Since \( \lambda = 0 \) is not an eigenvalue, we know

\[ \lambda u = 0 \]

has only \( u = 0 \) as the solution. This means

\[ \lambda u = f(x) \]

has a unique solution why? i.e. the SL two-point BVP is well-posed.
To prove \( \lambda \) are real we only need \( \lambda^* = \lambda \) and we proved this already.

To prove they are positive take

\[
(\eta, \lambda \eta) = \lambda (\eta, \eta)
\]

\[
= \int_a^b \left( -\eta \left( p \eta' \right)' \right) \text{ by parts}
\]

\[
= \int_a^b \left( p |\eta'|^2 + q |\eta|^2 \right) dx > 0
\]

+ vanishing boundary terms

\[
\Rightarrow \lambda = \frac{\int_a^b (p |\eta'|^2 + q |\eta|^2) dx}{\int |\eta|^2 dx}
\]

\[
\Rightarrow \lambda > 0
\]
We in fact know a few more things (not proven here):

1. The eigenvalues are simple (not repeated) and there are countably many of them with
   \[ 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \]
   and \( \lambda_k \to +\infty \) for \( k \to \infty \)

2. All eigenvectors are orthogonal (we have proven this)

3. The eigenvectors form a complete \( L^2 \) basis, i.e.,
   \[ f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \]
   convergence in norm
   \[ (u_n, f) \]