Lecture 12

Separation of Variables

In this first lecture of the second half of the semester we will motivate the rest by looking at the heat equation on a bounded domain:

\[ u_t = ku_{xx} \]

0 < x < 1, t > 0

\[ u(x, 0) = g(x) \quad \text{IC} \]

\[ u(0, t) = u(1, t) = 0 \quad \text{homogeneous Dirichlet BC} \]

Where we do not require that \( g(1) = g(0) = 1 \), allowing the solution to start discontinuous.
One way to solve this would be to find the Green's function by setting
\[ g(x) = \delta(x - x_0) \]
But this is almost as hard as the original problem because of the non-trivial BCs.

Let us try the ansatz (educated guess)

\[ u(x, t) = X(x) \, T(t) \]

Function only of \( x \) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad Function only of \( t \)

This means the two independent variables are separated.

It will take us several lectures to understand why this was a good guess.

But for now let's try it
BCs give:
\[
\begin{align*}
    u(0, t) &= \overline{x}(0) T(t) = 0 \\
    u(1, t) &= \overline{x}(1) T(t) = 1
\end{align*}
\]
This means (since \( T \neq 0 \))
\[
\overline{x}(0) = \overline{x}(1) = 0
\]
i.e. the spatial part satisfies the BCs.

New, plug into PDE
\[
\begin{align*}
    u_t &= \overline{x}(x) \overline{T}_t(t) \\
    ku_{xx} &= k \overline{x}_{xx}(x) \overline{T}_t(t)
\end{align*}
\]
\[
\Rightarrow \quad \frac{1}{k} \frac{\overline{T}'(t)}{\overline{T}(t)} = \frac{1}{\overline{x}(x)} \overline{x}''(x)
\]
Function of \( t \) Function of \( x \)

Since two functions of two different variables are equal, they must both be constant.
\[
\frac{1}{k} \frac{T'}{T} = \frac{X''}{X} = -\lambda
\]

Sign for later \(\lambda > 0\)

This is now a system of two ODEs!

The easier one to solve is

\[
T' = -\lambda k T \implies T(t) = T(0) \exp(-\lambda k t)
\]

and this exponential decay is typical of the heat equation

The harder equation is

\[
\begin{cases}
\overline{X}'' = -\lambda \overline{X} \\
\overline{X}(0) = \overline{X}(1) = 0
\end{cases}
\]

This is called a two-point boundary value problem
The trivial solution is \( 0 \)

but we want nontrivial ones.

The general solution has the form

\[ X = a e^{\lambda x} \]

where \( \lambda^2 = -\lambda \) from the ODE since the sign of \( \lambda \) is arbitrary.

\[ X = a e^{\lambda x} + b e^{-\lambda x} \]

is the general solution (we need two constants for second-order).

\[ X(0) = a + b = 0 \]

\[ X(1) = a e^{\lambda} + b e^{-\lambda} = 0 \]

\[ = a (e^{\lambda} - e^{-\lambda}) = 0 \]

If \( a = 0 \) \( \Rightarrow b = 0 \) so trivial.

If \( a \neq 0 \) then

\[ e^\lambda - \frac{1}{e^\lambda} = 0 \Rightarrow (e^\lambda)^2 = 1 \]
New let's assume that
\[ \lambda < 0 \]
\[ \lambda^2 = -\lambda > 0 \text{ and } \lambda > 0 \text{ is real} \]
\[ e^\lambda + 1 \Rightarrow \lambda = 0 \]
which is a contradiction

5. If \( \lambda = 0 \) then equation is

\[ \ddot{x} + 0 \Rightarrow \ddot{x} = ax + b \]
and \( \ddot{x}(0) = \ddot{x}(1) \) forces \( a = b = 0 \)
so we get the trivial solution

We conclude
\[ \lambda > 0 \]
\[ \lambda^2 = -\lambda < 0 \text{ so } \lambda = i\beta \text{ is complex} \]

\[ e^{i\beta x} = \cos(\beta x) + i\sin(\beta x) \]
and so it is better to switch now to sines and cosines
\[ X(x) = A \cos \beta x + B \sin \beta x \]

\[ X(0) = 0 \implies A = 0 \]

\[ X(1) = B \sin (\beta x) = 0 \implies \sin (\beta x) = 0 \]

Since \( B = 0 \) is trivial solution.

\[ \implies \beta = \text{integer, positive} \]

\[ \beta = n \frac{\pi}{\lambda} \in \mathbb{Z}^+ \]

\[ \lambda^2 = -\beta^2 = -\lambda \implies \lambda = n \frac{\pi}{\sqrt{11}} \text{ where } n \in \mathbb{Z}^+ \]

This is the fundamental result of the separation of variables.

The set of solutions of \( Xu = 0, \ B \bar{u} = 0 \)

is countable and can be enumerated by positive integers.
Sidetext: Compare this to the case of finite-dimensional $A\mathbf{x} = \mathbf{0}$, which has a finite number of independent solutions, called the nullity of the matrix.

\[ \mathbf{x}(x) = B \sin \left( \frac{n\pi x}{L} \right) \]

\[ \Rightarrow \]

\[ u(x,t) = B \mathbf{e}_n \sin \left( \frac{n\pi x}{L} \right) \]

We have therefore identified a family of countably infinitely many solutions.

Claim: Any solution can be expressed as a sum of these

\[ u(x,t) = \sum_{n=1}^{\infty} B_n \mathbf{e}_n \sin \left( \frac{n\pi x}{L} \right) \]

How to determine $B_n$?
The one thing we have not used is the initial condition:

\[ u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) = g(x) \]

The question now becomes:

For what functions \( g(x) \) does a convergent series of the above form exist? When it exists, is it unique, and if so, how to determine the \( B_n \) coefficients?

To answer this question, we will need to go into Fourier analysis.

It turns out almost every function can be expanded in this way and

\[ B_n = 2 \int_{-1}^{1} g(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]

as we will obtain later.
Some comments:
The problem
\[ \ddot{X} = -\lambda X, \quad X(0) = X(1) = 0 \]
is an eigenvalue problem very similar to the matrix \( AX = \lambda x \).
The \( \lambda \)'s that are possible are the eigenvalues and
\[ \overrightarrow{X}(x) = \sin(n\pi x) \]
are the eigenfunctions.

Many of the concepts from linear algebra will generalize, so we will review some of those in the upcoming lectures.

Also observe that the high-frequency modes (large \( n \)) decay very rapidly \( \exp(-kn^2\pi^2 t) \) and as \( t \) grows only the small \( n \)'s will survive and as \( t \to \infty \) the mode \( n=0 \) dominates.

→ smoothing property
Exercise for home

Change the BCs to homogeneous Neumann

\[ \dot{u}(0, t) = \dot{u}(1, t) = 0 \]

and show that the separable solution has the form

\[ u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\frac{k n^2}{2} t} \cos(n \pi x) \]

This is

\[ n=0 \]

satisfies BC

Now the IC becomes the equation for \( A_0 \) and \( A_n \):

\[ \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \left( n \pi \frac{x}{L} \right) = g(x) \]

Note that the same derivation can be generalized to the wave equation (and others)