Initial Value Problems

- We want to numerically approximate the solution to the **ordinary differential equation**

$$\frac{dx}{dt} = x'(t) = \dot{x}(t) = f\left[x(t), t\right],$$

with **initial condition** $x(t = 0) = x(0) = x_0$.

- This means that we want to generate an approximation to the **trajectory** $x(t)$, for example, a sequence $x(t_k = k\Delta t)$ for $k = 1, 2, \dots, N = T/\Delta t$, where Δt is the **time step** used to discretize time.
- If f is independent of t we call the system **autonomous**.
- Note that second-order equations can be written as a **system** of first-order equations:

$$\frac{d^2x}{dt^2} = \ddot{x}(t) = f\left[x(t), t\right] \quad \equiv \quad \begin{cases} \dot{x}(t) = & v(t) \\ \dot{v}(t) = & f\left[x(t), t\right] \end{cases}$$

Relation to Numerical Integration

- If f is independent of x then the problem is equivalent to numerical integration

$$x(t) = x_0 + \int_0^t f(s)ds.$$

- More generally, we cannot compute the integral because it depends on the unknown answer $x(t)$:

$$x(t) = x_0 + \int_0^t f[x(s), s]\, ds.$$

- Numerical methods are based on approximations of $f[x(s), s]$ into the "future" based on knowledge of $x(t)$ in the "past" and "present".

Euler's Method

- Assume that we have our approximation $x^{(k)}$ and want to move by one time step:

$$x^{(k+1)} \approx x^{(k)} + \int_{k\Delta t}^{(k+1)\Delta t} f\left[x(s), s\right] ds.$$

- The simplest possible thing is to use a piecewise constant approximation:

$$f\left[x(s), s\right] \approx f(x^{(k)}) = f^{(k)},$$

which gives the **forward Euler method**

$$x^{(k+1)} = x^{(k)} + f^{(k)}\Delta t.$$

- This method requires only one function evaluation per time step.

Global Error

- Now assume that the stability criterion is satisfied, and see what the error is at time T:

$$x^{(k)} - e^{\lambda T} = (1 + \lambda \Delta t)^{T/\Delta t} - e^{\lambda T} =$$
$$= \left(1 + \frac{\lambda T}{N}\right)^{N} - e^{\lambda T}.$$

- In the limit $N \to 0$ the first term converges to $e^{\lambda T}$ so the error is zero (the method converges).
- Furthermore, the correction terms are:

$$\left(1 + \frac{\lambda T}{N}\right)^{N} = e^{\lambda T}\left[1 - \frac{(\lambda T)^2}{2N} + O(N^{-2})\right]$$
$$= e^{\lambda T}\left[1 - \frac{\lambda^2 T}{2}\Delta t + O(\Delta t^2)\right],$$

which now shows that the relative error is $O(\Delta t)$ but generally grows with T.