Finite Difference Schemes (1)

for PARABOLIC EQS

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Consider the diffusion equation

$$\begin{cases} u_t = k\, u_{xx} \quad \text{on } [0,1] \\ u(x,0) = \eta(x) \\ u(0,t) = g_0(t) \\ u(1,t) = g_1(t) \end{cases}, \quad t > 0$$

Discretize in <u>space and time</u> using a finite difference interpretation

$x_i = ih, \quad i = 1, \dots, N$

$t_n = n\tau, \quad n = 1, 2, \dots$

Often we denote $h = \Delta x$, $\tau = \Delta t$

FD / $\boxed{U_i^n \approx u(x_i, t_n)}$ pointwise

METHOD OF LINES idea (MOL):

Discretize in space first to get a system of ODEs, then solve those ODEs using existing methods

NOTE: NOT ALL METHODS are MOL!
(especially HYPERBOLIC EQS)

SPATIAL DISCRETIZATION

the steady-state solution of a parabolic PDE satisfies an elliptic PDE:

$$u_{t\to\infty} = 0 \quad \Rightarrow \quad 0 = k\tilde{u}_{xx} + BCs$$

where $\quad u(x, t\to\infty) \longrightarrow \tilde{u}$

So the first step is to simply use a discretization of the elliptic operator, e.g., the 3^{pt} Laplacian

$$(u_{xx})_i \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

Denote this in matrix notation

$$u_{xx} \simeq A u$$

Now we get the <u>system of ODEs</u>

$$\boxed{\frac{du(t)}{dt} = kA\,u(t)}$$ $\boxed{\text{Set } k=1 \text{ for now}}$

where $u = \{u_1(t), \ldots, u_N(t)\}$

Now we can use any explicit, implicit, IMEX, or exponential method to solve these ODEs.

However, this system of ODEs has a <u>growing dimension</u> as we refine

Examples of popular discretizations (see HW5)

Forward Euler (explicit)

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{\Delta x^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)$$

or $$\boxed{u^{n+1} = u^n + (A u^n)\Delta t}$$

Crank-Nicolson or implicit midpoint

$$\frac{u^{n+1}-u^n}{\Delta t} = A\left(\frac{u^n+u^{n+1}}{2}\right)$$

$$\Rightarrow \boxed{\left(I - \frac{A}{2}\Delta t\right)u^{n+1} = \left(I + \frac{A\Delta t}{2}\right)u^n}$$

Local truncation error (LTE)

is done by Taylor series as usual

For Euler + 3 pt Laplacian:

$$\tau(x,t) = \frac{u(x,t+\tau) - u(x,t)}{\tau} - \frac{1}{h^2}\left(u(x-h,t) - 2u(x,t) + u(x+t,h)\right) =$$

$$\frac{1}{2}\tau u_{tt} + O(\tau^2) \leftarrow \text{temporal discretization error}$$

$$- \frac{h^2}{12} u_{xxxx} + O(h^4) \leftarrow \text{spatial error}$$

For $\boxed{\text{MOL error} = \text{spatial} + \text{temporal}}$

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$u_t = u_{xx}$ (very useful!) $\Rightarrow \begin{cases} u_{tt} = (u_{xx})_t = \\ = (u_t)_{xx} = u_{xxxx} \end{cases}$

$$\tau_{\text{Euler}}(x,t) = \left(\frac{\tau}{2} - \frac{h^2}{12}\right) u_{xxxx} + O(h^4 + \tau^2)$$

So the method is first-order in time and second order in space as expected.

<u>Crank-Nicolson</u> is second order in both space and time

$$\tau_{CN} = O(\tau^2 + h^2)$$

If we use $\boxed{\tau \sim h^2}$ for the (8)
Euler method, which as we will see shortly is <u>required</u> for stability, then

$$\boxed{\begin{aligned} \tau_{\text{Euler}} &= O(h^2) \quad \text{if} \quad \tau \sim h^2 \\ &\simeq \text{const.}\, u_{xxxx} \cdot h^2 \end{aligned}}$$

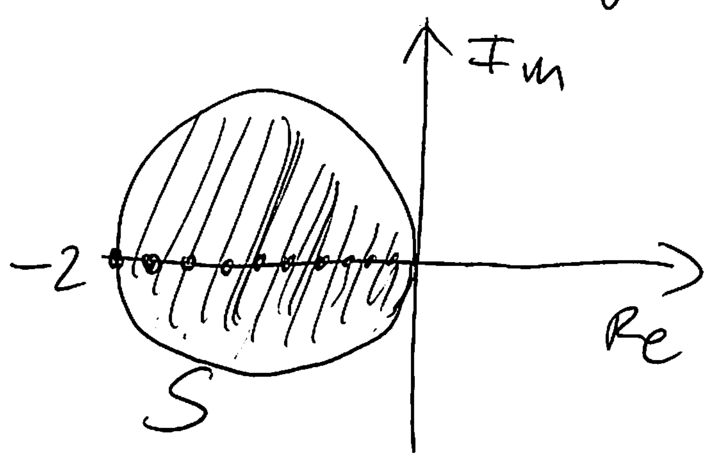
This will be true of <u>all explicit</u> methods for the diffusion equation:
We will require/need/choose $\tau \sim h^2$ and get second-order overall accuracy even with forward Euler

By contrast, for CN we want to choose $\boxed{\tau_{CN} \sim h}$ so that both spatial and temporal error are $O(h^2)$, i.e., neither one dominates.

When is Euler method A-stable?

We want $\lambda_p \Delta t \in S$ (stability region)

for all eigenvalues of A



Here all eigenvalues are <u>real and negative</u> (for Dirichlet BCs)

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$ (10)

$$\lambda_{mm} \approx -\frac{4}{h^2} \quad \text{when} \quad p\pi h \approx \pi$$

We want $|\lambda_{mm} \Delta t| \leq 2$

$$\Rightarrow \boxed{\frac{k\tau}{h^2} \leq \frac{1}{2}}$$

Courant - Friedrichs - Lewy

CFL condition for diffusion

Physics:

In time τ, diffusion spreads material over a distance $\sqrt{k\tau} \sim h$

Denote the CFL number

diffusive $\nu = \frac{k\tau}{h^2} \le \frac{1}{2}$

The LTE was

$$\tau_E \approx \left(\frac{k\tau}{2} - \frac{h^2}{12}\right) u_{xxxx}$$

$$= \left(\frac{\nu}{2} - \frac{1}{12}\right) h^2 u_{xxxx} = O(h^2)$$

The fact $\tau_E = O(h^2)$ is a problem → system of ODEs is stiff.

Crank-Nicolson is A-stable so no stability limit on τ, only accuracy & robustness (see HW5)

Convergence of FD methods

For a linear PDE the method must be linear and a one-step method will take the form

$$U^{n+1} = B(\tau)\, U^{n} + b^{n}(\tau)$$

e.g.

$$\begin{cases} B = \left(I - \frac{\tau}{2}A\right)^{-1}\left(I + \frac{\tau}{2}A\right) & \text{CN} \\ B = (I + \tau A) & \text{FE (Forward Euler)} \\ B = (I - \tau A)^{-1} & \text{Backward Euler} \end{cases}$$

A method is <u>LAX-RICHTMYER</u> stable if, for every $T>0$, $\forall\, C_T>0$ s.t.

$$\boxed{\|B(\tau)^n\| \le C_T}$$

for all $\tau>0$ and integers

$$n \le \frac{T}{\tau}$$

A fundamental result is the <u>Lax-Equivalence theorem</u>

$$\boxed{\text{A CONSISTENT METHOD is CONVERGENT } \underline{\text{IFF}} \text{ IT IS LAX-RICHTMYER STABLE}}$$

Proof of this is essentially identical as the proof that Euler's method converges for ODEs

$$E^n = U^n - \hat{U}^n \leftarrow \text{exact solution}$$

$$\boxed{E^{n+1} = B E^n - \tau (LTE)^n} \Rightarrow$$

$$\|E^N\| \le C_T \|E^0\| + T C_T \max_n |LTE|^n$$

If : $\boxed{\|B(\tau)\| \le 1 + \alpha\tau} \Rightarrow$

$$\|B^n\| \le (1+\alpha\tau)^{T/\tau} \le e^{\alpha T} = C_T$$

Often, we can prove or seek

$$\boxed{\|B\| \leq 1} \equiv \text{STRONG stability}$$ 15

E.g. Crank-Nicolson for diffusion:

Eigenvalues of B (symmetric) are

$$\lambda_B = \frac{1+\tau\lambda_A/2}{1-\tau\lambda_A/2}, \quad \lambda_A < 0$$

and $|\lambda_B| \leq 1$ for any $\tau > 0$

So that absolute stability ensures strong stability in the L_2 norm

Note that there is <u>no</u> concept of zero stability for PDEs like for ODEs, instead, we care about absolute stability.

This is because for PDEs as we add more points the eigenvalues can grow also, and so because τ and h are related it is often not true that $\tau \lambda \to 0$ as $\tau \to 0$.

Furthermore, while Euler's method could be shown to converge even for nonlinear ODEs, for PDEs there is only results for linear PDEs that are general

As we already discussed in class, with periodic BCs one can use Fourier series to <u>diagonalize</u> the FD matrices and thus obtain Eigenvalues and do stability analysis.

This is called $\boxed{\text{Von Neumann stability analysis}}$

<u>Read</u> section 9.6 of Le Veque

wavenumber

$$\hat{U}_k^n \simeq \hat{U}_k(n\Delta t), \quad U^n = \sum_k \hat{U}_k^n e^{ikx}$$

$$\boxed{\hat{U}_k^{n+1} = g(k)\,\hat{U}^n}, \quad k \in \left[0, \frac{\pi}{h}\right]$$

AMPLIFICATION FACTOR

$$\boxed{|g(k;\tau)| \leq 1 + \alpha k}$$

Observe that VON NEUMANN analysis (18)
is more general than just MOL schemes.
We don't have to rely on ODE theory

E.g. Euler for diffusion

In Fourier space symbol of Laplacian (3pt) is

$$\hat{L}_k = \frac{1}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right) = \frac{2}{h^2}\left(\cos(kh) - 1\right)$$

Euler's method

$$|g_k| = |1 + \tau \hat{L}_k| \leq 1$$

(strong stability)

$$\Rightarrow \quad 1 - \frac{4 \cdot \tau}{h^2} \leq 1 \quad \Rightarrow \quad \boxed{\tau \leq h^2/2}$$

as BEFORE

HOMEWORK: DO VON NEUMANN in TWO DIMENSIONS $u_t = \nabla^2 u$

How do we solve the <u>linear systems</u> that arise in implicit methods? E.g. Crank-Nicolson

$$\left(I - \frac{\tau}{2} A\right) x = b$$

Condition number

$$K_2^{CN} = \left| \frac{1 - \frac{\tau}{2}\lambda_{max}}{1 - \frac{\tau}{2}\lambda_{min}} \right|$$

$$K_2^{CN} \simeq \left| \frac{1 + \frac{8\tau}{2h^2}}{1 + \frac{\tau\pi^2}{2} \simeq 1} \right| \simeq \left| 1 + 4\nu \right|$$

Condition number is <u>bounded</u> $\therefore$ $\boxed{K_2^{CN} \simeq O(\nu)}$

So now $\boxed{\text{iterative methods are OK if } \boxed{\nu \gg 1}}$!

$\boxed{\text{Use } u^n \text{ as an initial guess}}$