Finite Difference & Volume for HYPERBOLIC PDEs

(1)

A. DONEV, COURANT

Hyperbolic PDEs are much harder than parabolic especially non-linear ones, and the right methods for them are FINITE VOLUME (FV) methods.

However, to second order accuracy ~~there~~ there is no difference between finite difference/volume except for boundary conditions, so we start with FD

As an example we consider the <u>advection equation</u>

$$u_t + (a(x)\,u)_x = 0$$

$$u_t = -\frac{d}{dx} f(u; x, t)$$

advective <u>flux</u>

where $f = a(x)\,u$

In higher dimensions

<u>CONSERVATION LAW</u>

$$\boxed{\begin{aligned} u_t &= -\nabla \cdot (\vec{a}(x,t)\,u) \\ &= -\nabla \cdot (\text{flux}) \end{aligned}}$$

Let's start with unbounded / periodic domains first and ignore boundary conditions for now.

START WITH : $\boxed{a(x) = \text{const} = a}$ or $\nabla \cdot \vec{a} = 0$ in higher dims

$$\boxed{\begin{aligned} u_t + a u_x &= 0 \\ u(x,0) &= \eta(x) \end{aligned}}$$

$u_t + \vec{a} \cdot \vec{\nabla} u = 0$ in 2D/3D

This is a trivial equation since the solution simply translates with velocity a, to the right if $a>0$ or to the left if $a<0$

$$u(x,t) = \eta\,(x-at)$$

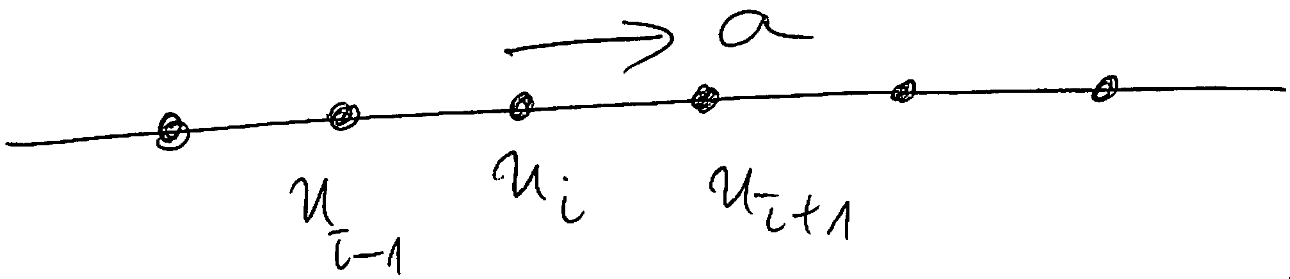
is the solution

(in periodic or unbounded)

More generally, we use the method of characteristics to solve these equations, both on paper and on a computer.

Even though this equation is trivial, numerical methods to solve it are hard to construct!

Start with finite difference



Imagine that the time step size is

$$\tau = \frac{h}{a}$$

Then, the exact solution is

$$u_{i+1} = u_i \quad !$$

But this only works in this trivial case, and for a fixed τ.

Why is the advection equation harder?
than diffusion

In Fourier space

$$\hat{u}_t = -iak\,\hat{u}$$

So the eigenvalues are <u>PURELY IMAGINARY</u>

$$\boxed{\lambda_k = -iak}$$

This means that the PDE is <u>non-dissipative</u>, the L_2 norm should remain constant for <u>all</u> modes k, just like for the KdV equation.

But numerical error will either introduce artificial growth or decay of modes, or be unstable

Let's try a simple FD method:

$$u_t = -a u_x \Rightarrow \text{MOL scheme:}$$

$$\underbrace{\frac{U_j^{n+1} - U_j^{n}}{\tau}}_{\text{Forward Euler}} = \frac{a}{2h} \underbrace{\left(U_{j+1}^{n} - U_{j-1}^{n} \right)}_{\text{Centered FD}}$$

$$U^{n+1} = U^{n} + \tau A U^{n}$$

$$A = -\frac{a}{2h} \begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ 1 & & & & -1 & 0 \end{bmatrix}$$

periodic

$$U^{n+1} = B U^n$$

$$B = I + \tau A$$

We need $\|B\| \leq 1 + \alpha \tau$ for Lax-Richtmyer stability. Let's use l_2 norm. The eigenvalues of A can be found by going to Fourier space

Eigenvector $u_j^p = e^{i \frac{2\pi}{L} p j h}$

where $0 \leq x < L$

Plug into the stencil for A to get symbol of centered difference

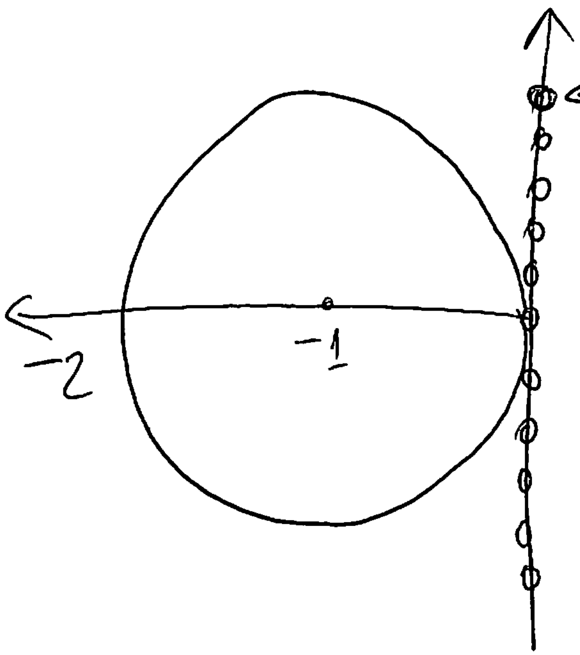
$$\lambda_p = -\frac{a}{2h}\left(e^{i\frac{2\pi}{L}p} - e^{-i\frac{2\pi}{L}p}\right)$$

$$\Rightarrow \quad \lambda_p = -\frac{ia}{h}\sin\left(\frac{2\pi}{L}ph\right)$$

$$\simeq \underbrace{-ia\left(\frac{2\pi}{L}p\right)}_{-iak \,=\, \text{truth}} + O(h^2)$$

$$\boxed{\text{Eigenvalues for centered difference are all purely imaginary}}$$

But stability region for Euler <u>does NOT</u> include a finite portion of the imaginary axis!



Largest modulus is for $k = \frac{\pi}{2h}$

$$|\lambda_{max}| = \left|\frac{ia}{h}\sin(kh)\right| = \frac{a}{h}$$

$$\boxed{-\frac{ia}{h} \le \lambda_p \le \frac{ia}{h}}$$

$$|1+\tau\lambda|^2 \le 1 + \tau^2|\lambda_{max}|^2$$
$$= 1 + \frac{\tau^2 a^2}{h^2}$$

$$\Rightarrow \|I + kA\|_2^2 = \|B\|_2^2 \leq 1 + \left(\frac{\tau a}{h}\right)^2 \qquad (11)$$

What we want is $\|B_2\|^2 \leq 1 + \alpha\tau$

If we take $\tau \sim h$ then

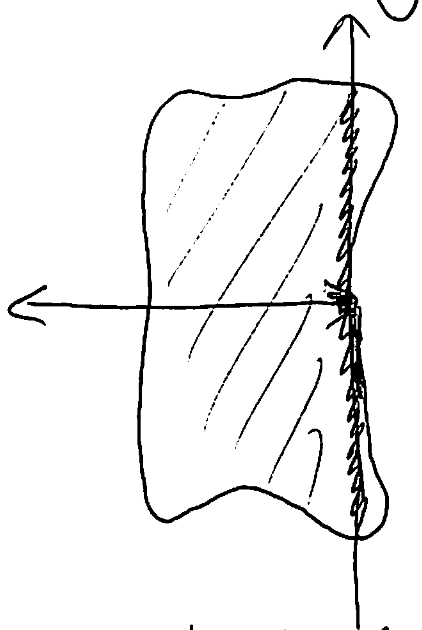
$$\|B\|_2^2 \leq 1 + \text{const} \quad \times \text{ Does not work}$$

If we take $\tau \sim h^2$ then

$$\|B\|_2^2 \leq 1 + \text{const} \cdot h^2 = 1 + \alpha\tau \quad \checkmark$$

$\boxed{\text{So to get convergence with Euler's method + CENTERED difference we need } \tau \sim h^2 \text{ - BAD}}$

Instead, let's consider using something like RK3, whose absolute stability region includes the interval



$$[-ic, ic]$$

for some $c > 0$, $c \simeq O(1)$

So now we can get strong stability if

$$|\tau \lambda_{max}| \le c$$

$$\frac{\tau a}{h} \le c \Rightarrow \boxed{\tau \le c \frac{h}{a} \quad c \approx 1}$$

This is a natural choice for the advection equation and all hyperbolic equations. (13)

[Information propagates along characteristics with speed a

So $\boxed{\tau \sim h/a}$ means that information does not propagate further than a grid spacing, which is required for a stencil with only 3 points

So centered differencing + Euler is a bad idea for advection.

At the same time, RK3 is <u>overkill</u>. It is expensive, and we get temporal error of order (14)

$\tau^3 \sim h^3$ but spatial error of $O(h^2)$

so they are not balanced.

But RK2 is also unstable for imaginary eigenvalues!

<u>Advection is</u> not as stiff as diffusion, in fact, we can consider it <u>not stiff</u> because high frequency modes ($|k|$ large) are <u>not</u> damped so they matter $\Rightarrow$ EXPLICIT methods are <u>OK</u>

So we need to consider other explicit methods. (15)

Let's try for first-order

$$U_j^{n+1} = U_j^n - \frac{a\tau}{h}\left(U_j^n - U_{j-1}^n\right)$$

First order in space and time

Performing von Neumann analysis, in Fourier space we get

$$\hat{u}_k^{n+1} = \hat{u}_k^n - \frac{a\tau}{h}\left(e^0 - e^{-ikh}\right)\hat{u}_k^n$$

$$\hat{U}_k^{n+1} = g_k(\tau)\,\hat{U}_k^n$$

Denote the ADVECTIVE CFL number

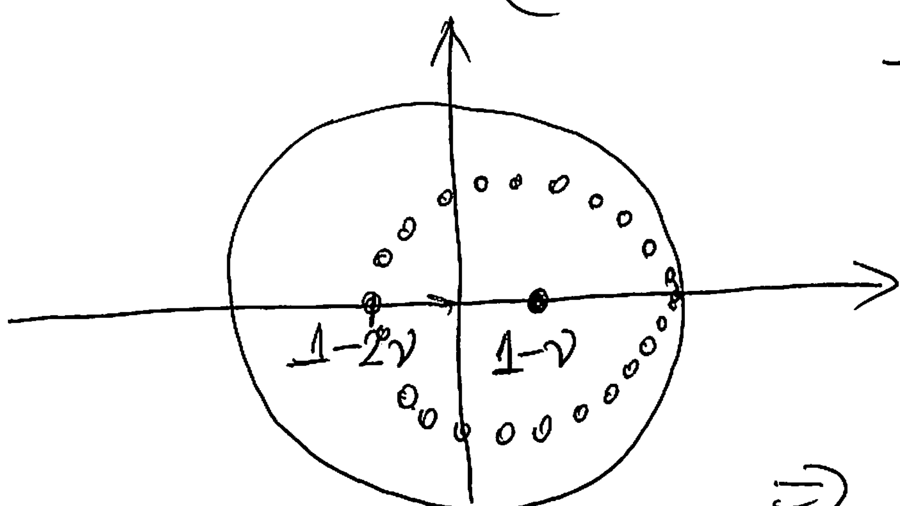
$$\boxed{\nu = \frac{a\tau}{h}}$$

The amplification factor is

$$g_k = 1 - \nu\left(1 - e^{-ikh}\right)$$

$$= (1-\nu) + \nu e^{-ikh}$$

$$-\frac{\pi}{h} \le k \le -\frac{\pi}{h}$$



We want

$$|g_k| \le 1$$

$$\Rightarrow \quad 1 - 2\nu \ge -1, \quad \nu \ge 0$$

So we get strong stability if $\boxed{0 \leq \nu \leq 1}$ so $\boxed{a \geq 0}$

Observe that for $\nu = 1$ the scheme gives

$$U_j^{n+1} = U_j^n = \underline{\underline{\text{exact}}}\ !$$

To get a stable scheme we do

$$\boxed{\underset{\text{UPWIND}}{\underline{}}\quad \frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{h}\begin{cases} u_j^n - u_{j-1}^n, & a \geq 0 \\ u_{j+1}^n - u_j^n, & a \leq 0 \end{cases}}$$

This is the first-order **upwind scheme**: If information propagates to the right, we use the points to the left, otherwise on the right.
This is very obvious physically.
It also generalizes naturally to complex hyperbolic equations:

Follow the characteristics backward in time to $t = -\tau$ and use that to compute solution now at $t = 0$

With the choice :

$$0 \le \nu = \text{const} \sim O(1) \le 1$$

$$\tau \sim h$$

the upwind scheme is first order in both space and time and overall.

It is a very inaccurate scheme except for $a = \text{const}$ and $\nu = 1$

We really want something at least second-order accurate in space and time, stable for $\tau \sim h$!

Taylor series in time:

$$u(t+\tau) = u(t) + u_t \tau + \frac{1}{2} u_{tt} \tau^2 + O(\tau^3)$$

$$u_t = -a u_x \Rightarrow u_{tt} = -a (u_t)_x$$

$$u_{tt} = +a^2 u_{xx}$$

So we get

$$\boxed{u(t+\tau) \simeq u(t) - a u_x \tau + \frac{1}{2} a^2 \tau^2 u_{xx}}$$

artificially dissipative term with diffusion coeff $a^2\tau^2/2$

Now we can discretize this Taylor series in space to get the <u>second-order</u> <u>Lax-Wendroff</u>:

$$U_j^{n+1} = U_j^n - \frac{a\tau}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2\tau^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

i.e. in matrix notation

$$\boxed{\underline{U}^{n+1} = U^n - \frac{a\tau}{h} D_0 U^n + \frac{a^2\tau^2}{2h^2} D^2 U^n}$$

centered deriv. centered 2^{nd} der.

It is obvious that this is second-order from its discretization & derivation.

Is it stable for $\tau \sim h$?

Observe that the Lax-Wendroff (LW) method is NOT an MOL method

If we discretize in space as

$$\frac{dU}{dt} = AU$$

then any method will only generate rational functions of A

So a Taylor series / RK2 method would give

$$U^{n+1} = U^n + \tau A u^n + \frac{\tau^2 A^2}{2} U^n$$

But here $A = -\frac{a}{h} D_0$ so

$$\frac{A^2}{2} = \frac{a^2}{2h^2} D_0^2 \neq \frac{a^2}{2h^2} D^2$$

because as you may recall

$$(D_0^2 u)_j = \frac{1}{4h^2}\left(U^n_{j-2} - 2u_j + u^n_{j+2}\right)$$

BAD IDEA! <u>Wide</u> 5 pt stencil

Since not an MOL line, we cannot (should not, LeVeque does it but only works for Euler) use ODE stability regions.

But we can still <u>use von-Neumann</u>.

Amplification factor

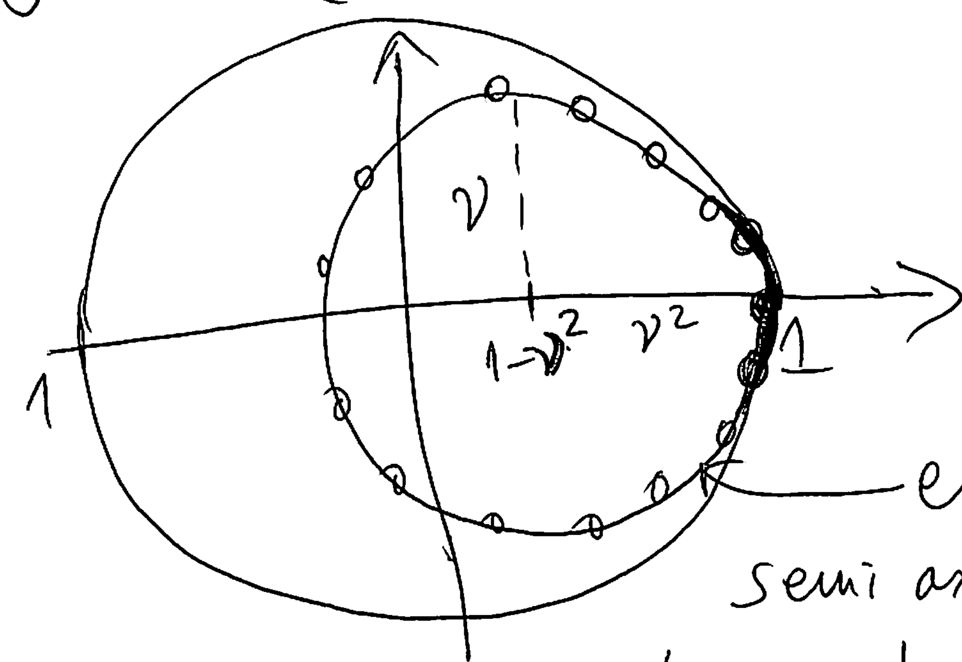
$$g_k = 1 - \frac{\nu}{2}\left(e^{ikh} - e^{-ikh}\right) + \frac{\nu^2}{2}\left(e^{ikh} - 2 + e^{-ikh}\right)$$

+ algebra ....
(Example 10.3 in LeVeque)

$$g_k = \left(1 - \nu^2(1-\cos\theta)\right) - i\nu\sin\theta \quad , \quad \theta = kh$$

$$\boxed{-\pi \le \theta \le \pi}$$

$$g_k = (1-\nu^2) + \nu\,(\nu\cos\theta - i\sin\theta)$$



we want

$$|g_k| \le 1$$

ellipse with semi axes $|\nu|$ and ν^2

<u>tangent</u> to the unit circle

The reason we get stability for $\boxed{|\nu| \leq 1}$ for LW is that the apparent diffusive term in the Taylor series brings dissipation which adds a small negative real part to the eigenvalues.

Lax-Wendroff adds the smallest possible amount of dissipation/diffusion to centered advection to stabilize forward Euler

Observe that upwinding also adds artificial dissipation to centered advection since we can rewrite the upwind scheme as

$$U_j^{n+1} = U_j^n - \frac{a\tau}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a\tau}{2h}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

Artificial dissipation is $O(\tau)$ and not $O(\tau^2)$ so this scheme smears solutions a lot

One can also make one-sided or
upwind biased second-order schemes.

An example is the **Beam-Warming** scheme: $\boxed{a>0}$

$$U_j^{n+1} = U_j^n - \frac{a\tau}{2h}\Big(\underbrace{3U_j^n - 4U_{j-1}^n + U_{j-2}^n}_{\text{second-order one-sided finite difference } u'(x)}\Big)$$

$$\underbrace{+ \frac{a^2\tau^2}{2h^2}\left(U_j^n - 2U_{j+1}^n + U_{j+2}^n\right)}_{\text{second-order in time like LW}}$$

One can derive the Beam warming scheme and Lax-Wendroff as semi-Lagrangian methods for advection, see 10.6 in LeVeque "Characteristic tracing & interpolation"

But this is specific for the advection scheme and not so general so I will skip it.

Note Beam-Warming is a SPACE-TIME SCHEME like LW NOT MOL

A related MOL method is the 3rd-order upwind biased method

30

$\mathrm{Re}(\lambda_p) < 0$!

$$u_j' = \frac{a}{h}\left[-\frac{1}{6}u_{j-2} + u_{j-1} - \frac{1}{2}u_j - \frac{1}{3}u_{j+1}\right]$$

MOL scheme → combine with RK3

which is better than the 4th order centered advection (non-dissipative)

$$u_j' = \frac{a}{h}\left[-\frac{1}{12}u_{j-2} + \frac{2}{3}u_{j-1} - \frac{2}{3}u_{j+1} + \frac{1}{12}u_{j+2}\right]$$

All eigenvalues imaginary