A. Popov

Lecture 4

(1)

# MARKOV PROCESSES

Let $\{X_t\}$ be a homogeneous (time invariant) Markov process. Define the evolution operator

$$P_t f(x) = \mathbb{E}\left[ f(X_t) \mid X_0 = x \right]$$ ← semigroup

If the strong limit

$$\mathcal{L} f = \lim_{\Delta t \to 0} \frac{(P_{\Delta t} f) - f}{\Delta t}$$

exists, we call $\mathcal{L}$ the infinitesimal <u>generator</u> of the Markov process.

It fully characterizes it.

P.001

SEP-28-2011 10:20

At least formally, if $\mathcal{L}$ does not depend on time (homogeneous), we write:

$$P_t = \exp\left[\mathcal{L} t\right]$$

Consider the evolution of an observable

$$u(x,t) = \mathbb{E}\left[u(X_t) \mid X_0 = x\right]$$

abuse notation

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \mathcal{L} u, \quad u(x,0) = u_0(x)}$$

Backward Kolmogorov equation

Now consider the evolution of a <u>state</u>, more precisely, a probability measure $\mu$ of initial conditions in phase space. For simplicity, assume that this measure can be related to a probability distribution function $\rho(y,t)$:

$$\mu_t = \rho(y,t)\, dy$$

which Mathematicians call the <u>law</u> of the Markov process.

The time evolution of $\rho(y,t)$ is governed by the Fokker-Planck equation

$$\boxed{\frac{\partial \rho}{\partial t} = \mathcal{L}^{*} \rho \qquad \text{Forward Kolmogorov eq.}}$$

where $\mathcal{L}^{*}$ is the $L^2$-adjoint of the generator

$$\underbrace{\int (\mathcal{L} f)\, h \, dx}_{\substack{\text{expectation of } f \\ \text{if initial conditions} \\ \text{are distributed according} \\ \text{to } h}} = \int f \underbrace{(\mathcal{L}^{*} h)\, dx}_{\substack{\mathcal{L}^{*} h \text{ is the} \\ \text{time-evolved} \\ \text{probability distribution}}}$$

A Markov process is ergodic if the generator $\mathcal{L}$ has only constants in its null space.

An ergodic Markov process has an <u>invariant measure</u> or <u>invariant density</u>

$$\mu_{eq}(dx) = \rho_{eq}(x)\,dx$$

$$\boxed{\mathcal{L}^* \rho_{eq} = 0}$$

which is unique since $\mathcal{L}$ is ergodic

$$\boxed{\text{Time averages of trajectories equal phase space averages w.r.t } \rho_{eq}}$$ (ergodic)

## Examples:

) Recall Hamiltonian dynamics:

$$\partial_t \rho = - \underbrace{iL}_{\text{Louiville operator}} \rho$$

$$(iL)^* = -iL^* = +iL$$

$$\Rightarrow \quad \partial_t f = iL f = \dot{z} \frac{\partial f}{\partial z}$$

2) For jump Markov processes on a finite state space, $\mathcal{L}$ is just the matrix of transition probabilities

$$\mathcal{L}_{ij} = w_{i \to j}, \quad \mathcal{L}_{ii} = -\sum_j w_{i \to j}$$

Note that

$$\mathcal{L} \cdot \mathbb{1} = 0$$

vector of ones in null space

and $\mathcal{L}^* \pi = 0 \Rightarrow$

$$\forall i: \quad \mathcal{L}_{ji} \pi_j = \omega_{ji} \pi_j = 0$$

$$\Rightarrow \quad \omega_{ji} \pi_j + \pi_i = \pi_i$$

Discrete time Markov chain

$$\begin{cases} P_{ij} \pi_j = \pi_i \\ \text{where} \quad P_{ii} = 1 - \sum_j \omega_{ij} \end{cases}$$

(3) Stochastic Differential equation:

$$\dot{x} = \underset{\substack{\uparrow \\ \text{drift}}}{h}(x) + \underset{\substack{\uparrow \\ \text{diffusion}}}{g}(x)\, \mathcal{W}(t)$$

The FPE is the forward Kolmogorov

$$\frac{\partial P}{\partial t} = \mathcal{L}^* P = -\frac{\partial}{\partial x} \cdot (h P) + \frac{1}{2} \frac{\partial^2}{\partial x^2} : (g g^* P)$$

$$\Rightarrow \mathcal{L}^* = -\frac{\partial}{\partial x} \cdot h + \frac{1}{2} \frac{\partial^2}{\partial x^2} : (g g^*)$$

How do we take the adjoint of this to get the generator $\mathcal{L}$?

Here is a quick-and-dirty way:

Adjoint of a gradient is negative divergence:

$$\left(\frac{\partial}{\partial x}\right)^* = -\frac{\partial}{\partial x}\cdot$$ (think integration by parts)

$$\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x} = \left(\frac{\partial^2}{\partial x}:\right)^* = \left(\frac{\partial^2}{\partial x^2}:\right)$$ (Laplacian is self adjoint),

$\Rightarrow$

$$\mathcal{L} = \underbrace{h\cdot\frac{\partial}{\partial x} + \frac{1}{2}(gg^*):\frac{\partial^2}{\partial x^2}}$$

This is Ito's formula for the drift in $f(x(t))$