Fluctuating Burgers and Navier-Stokes Equations

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1 Introduction

Recall the augmented Langevin equation in the canonical (isothermal) ensemble,

$$\partial_t \boldsymbol{x} = -\boldsymbol{N}(\boldsymbol{x}) \frac{\partial H}{\partial \boldsymbol{x}} + (2k_B T)^{1/2} \boldsymbol{B}(\boldsymbol{x}) \, \boldsymbol{\mathcal{W}}(t) + (k_B T) \frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{N}^\star(\boldsymbol{x}), \tag{1}$$

which preserves the Gibbs-Boltzmann distribution

$$P_{\text{eq}}(\boldsymbol{x}) = Z^{-1} \exp\left[-\frac{H(\boldsymbol{x})}{k_B T}\right]. \tag{2}$$

The skew-adjoint operator

$$\boldsymbol{S} = -\boldsymbol{S}^\star = \frac{1}{2}\left(\boldsymbol{N}^\star - \boldsymbol{N}\right)$$

generates the "conservative" part of the dynamics, and the self-adjoint positive semi-definite operator

$$\boldsymbol{M} = \boldsymbol{M}^\star = \frac{1}{2}\left(\boldsymbol{N} + \boldsymbol{N}^\star\right) \succeq \boldsymbol{0}$$

generates the "dissipative" part of the dynamics. The operator $\boldsymbol{B}(\boldsymbol{x})$ is constrained by the *fluctuation-dissipation balance* condition

$$\boldsymbol{B}\boldsymbol{B}^\star = \boldsymbol{M}.$$

2 Fluctuating Burgers Equation

We consider the *fluctuating Burgers equation* for the random field $u(x,t)$,

$$\partial_t u + cu\,\partial_x u = \nu\,\partial^2_{xx} u + (2\nu)^{\frac{1}{2}}\,\partial_x \mathcal{Z}, \tag{3}$$

where ν is a diffusion coefficient and c sets the scale for the advection speed. An equivalent conservative form is

$$\partial_t u = -\partial_x \left[c\frac{u^2}{2} - \nu\partial_x u - (2\nu)^{\frac{1}{2}}\,\mathcal{Z} \right].$$

This equation obeys a fluctuation-dissipation balance principle with respect to the Gibbs-Boltzmann distribution with a Hamiltonian

$$H\left[u\left(x,t\right)\right] = \int \frac{u^2}{2} dx, \tag{4}$$

so that

$$\frac{\partial H}{\partial u} \equiv \frac{\delta H\left[u\left(x,t\right)\right]}{\delta u} = u.$$

The dissipative and fluctuating dynamics in (3) are generated by the constant operators,

$$\boldsymbol{M} = -\nu\partial^2_{xx} \text{ and } \boldsymbol{B} = \nu^{\frac{1}{2}}\partial_x,$$

which in higher dimensions become multiples of the Laplacian and divergence operators, respectively. The conservative dynamics for the Burgers equation is Hamiltonian and generated by the skew-adjoint linear operator $\boldsymbol{S}(u)$ defined through its action on a field $w(x,t)$ [1],

$$\boldsymbol{S}\left(u\right)\,w = -\frac{c}{3}\left[u\partial_x w + \partial_x\left(uw\right)\right]. \tag{5}$$

An important property of Hamiltonian dynamics is that it is incompressible in phase space,

$$\frac{\partial}{\partial u} \cdot \boldsymbol{S}\left(u\right) = \frac{\partial}{\partial u} \cdot \boldsymbol{N}^{\star}\left(u\right) = \boldsymbol{0}. \tag{6}$$

This implies that the dynamics of the inviscid Burgers equation preserves not just functions (such as the Hamiltonian itself) but also phase-space measures (such as the Gibbs distribution), and thus any probability density that is a function of H only is a candidate equilibrium distribution.

We now discuss spatial discretization of the fluctuating Burgers equation. The discretized $\boldsymbol{u} = \{u_1, \ldots, u_N\}$ can be thought of as a finite-volume representation of the field $u(x,t)$ on a regular grid with spacing Δx, specifically, u_j can be thought of as representing the average value of $u(x,t)$ over the interval (cell) $[j\Delta x,\,(j+1)\Delta x]$. Similarly, the spatially-discretized collection of white noise processes $(\Delta x)^{-1/2}\,\boldsymbol{\mathcal{W}}$ can formally be associated with the space-time white noise $\boldsymbol{\mathcal{Z}}$.

We take the coarse-grained Hamiltonian function to be the natural (local equilibrium [2]) discretization of (4),

$$H\left(\boldsymbol{u}\right) = \sum_{j=1}^{N} \frac{\Delta x}{2} u_j^2, \tag{7}$$

We will construct a spatial discretization that leads to a finite-dimensional generic Langevin equation of the form (1)

$$\partial_t \boldsymbol{u} = \boldsymbol{S}\frac{\partial H}{\partial \boldsymbol{u}} + \frac{\nu}{\Delta x}\boldsymbol{D}_2\frac{\partial H}{\partial \boldsymbol{u}} + \left(\frac{2\nu}{\Delta x}\right)^{1/2} \boldsymbol{D}_1 \boldsymbol{\mathcal{W}}(t). \tag{8}$$

Here $\boldsymbol{\mathcal{W}}$ is a vector of N_w independent white-noise processes (formally, time derivatives of independent Wiener processes), $\boldsymbol{D}_1$ is a matrix representing the spatial discretization of the divergence operator, such that $\boldsymbol{D}_2 = -\boldsymbol{D}_1\boldsymbol{D}_1^{\star}$ is a symmetric negative-semidefinite discretization of the Laplacian operator. This system of SODEs has as an invariant distribution the Gibbs distribution (2) if $\boldsymbol{S}$ is an antisymmetric matrix discretizing (5) that satisfies

$$\left[\frac{\partial}{\partial \boldsymbol{u}} \cdot \boldsymbol{S}\left(\boldsymbol{u}\right)\right]_k = \sum_j \frac{\partial S_{j,k}}{\partial u_j} = 0 \text{ for all } k. \tag{9}$$

We now construct specific finite-difference operators for $\boldsymbol{D}_1$ and $\boldsymbol{S}$.

A particularly simple choice that also generalizes to higher dimensions [3] is to associate fluxes with the half-grid points (faces of the grid in higher dimensions), and to define

$$\left(\boldsymbol{D}_1\boldsymbol{\mathcal{W}}\right)_j = \frac{\mathcal{W}_{j+\frac{1}{2}} - \mathcal{W}_{j-\frac{1}{2}}}{\Delta x}, \text{ giving } \left(\boldsymbol{D}_1^{\star}\boldsymbol{u}\right)_{j+\frac{1}{2}} = -\frac{u_{j+1} - u_j}{\Delta x}.$$

This construction gives the familiar three-point discrete Laplacian ($2d + 1$ points in dimension d),

$$\left(\boldsymbol{D}_2\boldsymbol{u}\right)_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2}, \tag{10}$$

and is therefore an attractive choice that satisfies the discrete fluctuation-dissipation principle [3]. If periodic boundary conditions are imposed, we set $u_0 = u_N$ and $u_{N+1} = u_1$ and $\mathcal{W}_{\frac{1}{2}} = \mathcal{W}_{N+\frac{1}{2}}$ (i.e., $N_w = N$). For Dirichlet boundary conditions we fix u_0 and u_{N+1} at specified values and do not need to impose any boundary conditions on $\mathcal{W}$ (i.e., $N_w = N + 1$).

A natural choice for $\boldsymbol{S}$ is formed by choosing a skew-adjoint discretization $\widetilde{\boldsymbol{D}}_1 = -\widetilde{\boldsymbol{D}}_1^{\star}$ of ∂_x, in general different from $\boldsymbol{D}_1$, and discretizing (5) directly as

$$\left(\boldsymbol{S}\boldsymbol{u}\right)_j = -\frac{c}{3}\left[u_j\left(\widetilde{\boldsymbol{D}}_1 u\right)_j - \left(\widetilde{\boldsymbol{D}}_1^{\star} u^2\right)_j\right] = -\frac{c}{3}\left[u_j\left(\widetilde{\boldsymbol{D}}_1 u\right)_j + \left(\widetilde{\boldsymbol{D}}_1 u^2\right)_j\right],$$

where $u^2 = \{u_1^2, \ldots, u_N^2\}$. We choose $\widetilde{\boldsymbol{D}}_1$ to be the second-order centered difference operator

$$\left(\widetilde{\boldsymbol{D}}_1\boldsymbol{u}\right)_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x},$$

leading to an explicit expression that makes it clear that $\boldsymbol{S}\boldsymbol{u}$ is a discretization of $-cuu_x$,

$$\begin{aligned}(\boldsymbol{S}\boldsymbol{u})_j &= -\frac{c}{3}\left[u_j\left(\frac{u_{j+1}-u_{j-1}}{2\Delta x}\right)+\frac{u_{j+1}^2-u_{j-1}^2}{2\Delta x}\right]= \\ &= -c\left(\frac{u_{j-1}+u_j+u_{j+1}}{3}\right)\left(\frac{u_{j+1}-u_{j-1}}{2\Delta x}\right).\end{aligned}$$

It is important to note that one can write the nonlinear term in conservative form,

$$(\boldsymbol{S}\boldsymbol{u})_j = -\frac{c}{2}\left(\frac{u_{j+\frac{1}{2}}^2-u_{j-\frac{1}{2}}^2}{\Delta x}\right), \text{ where } u_{j+\frac{1}{2}}^2 = \frac{u_j^2+u_ju_{j+1}+u_{j+1}^2}{3}. \tag{11}$$

Due to the skew-symmetry, in the absence of viscosity the total "energy" (7) is conserved for periodic systems. It can also easily be shown that that the condition (6) is satisfied and therefore this particular discretization of the advective term preserves the Hamiltonian structure of the equations [4].

Putting the pieces together we can write the semi-discrete fluctuating Burgers equation as a system of SODEs, $j = 1, \ldots, N$,

$$\begin{aligned}\frac{du_j}{dt} &= -\frac{c}{6\Delta x}\left(u_{j-1}+u_j+u_{j+1}\right)\left(u_{j+1}-u_{j-1}\right) \qquad (12)\\ &+ \frac{\nu}{\Delta x^2}\left(u_{j-1}-2u_j+u_{j+1}\right)+\frac{(2\nu)^{1/2}}{\Delta x^{3/2}}\left(\mathcal{W}_{j+\frac{1}{2}}(t)-\mathcal{W}_{j-\frac{1}{2}}(t)\right).\end{aligned}$$

With periodic boundary conditions, this stochastic method of lines [5] discretization strictly conserves the total energy (7) and the total momentum

$$m(\boldsymbol{u}) = \sum_{j=1}^{N}\Delta x\, u_j.$$

The equilibrium distribution is the discrete Gibbs-Boltzmann distribution

$$P_{\text{eq}}\left(\boldsymbol{u}\right) = Z^{-1} \exp\left[-\frac{\Delta x}{2} \sum_{j=1}^{N} u_j^2\right] \delta\left(\Delta x \sum_{j=1}^{N} u_j - m_0\right),$$

where m_0 is the initial value for the total momentum.

3 Fluctuating Navier-Stokes Equation

The prototype stochastic partial differential equation (SPDE) of fluctuating hydrodynamics is the fluctuating Navier-Stokes equation. This equation approximates the dynamics of the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ of a simple Newtonian fluid in the isothermal and incompressible approximation, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$,

$$\rho\left(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = -\boldsymbol{\nabla}\pi + \eta \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla} \cdot \left[\left(k_B T \eta\right)^{1/2} \left(\boldsymbol{\mathcal{Z}} + \boldsymbol{\mathcal{Z}}^T\right)\right] \tag{13}$$

where π is the non-thermodynamic pressure, ρ is the (constant) density, $\eta = \rho\nu$ is the (constant) shear viscosity and ν is the kinematic viscosity, and $\boldsymbol{f}\left(\boldsymbol{r}, t\right)$ is an additional force density such as gravity [6]. The stochastic momentum flux is modeled using a white-noise random Gaussian tensor field $\boldsymbol{\mathcal{Z}}\left(\boldsymbol{r}, t\right)$, that is, a tensor field whose components are independent (space-time) white noise processes,

$$\langle \mathcal{Z}_{ij}(\boldsymbol{r}, t)\mathcal{Z}_{kl}(\boldsymbol{r}', t')\rangle = \left(\delta_{ik}\delta_{jl}\right)\delta(t - t')\delta(\boldsymbol{r} - \boldsymbol{r}').$$

At thermodynamic equilibrium, the invariant measure (equilibrium distribution) for the fluctuating velocities with periodic boundaries is the Gibbs-Boltzmann distribution with a coarse-grained free energy or Hamiltonian given by the kinetic energy of the fluid, formally,

$$P_{\text{eq}}\left(\boldsymbol{v}\right) = Z^{-1} \exp\left[-\frac{\int d\boldsymbol{r}\, \rho v^2}{2k_B T}\right] \boldsymbol{\delta}\left(\int d\boldsymbol{r}\, \rho \boldsymbol{v}\right) \delta\left(\boldsymbol{\nabla} \cdot \boldsymbol{v}\right).$$

Eliminate pressure from (13) using a projector formalism,

$$\partial_t \boldsymbol{v} = \boldsymbol{\mathcal{P}} \left[-\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} + \left(2\nu\rho^{-1}\, k_B T \right)^{\frac{1}{2}} \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{Z}}_{\boldsymbol{v}} \right]. \tag{14}$$

The divergence-free constraint is a constant linear constraint and the projection restricts the velocity dynamics to the constant linear subspace of divergence-free vector fields. The fluctuations in the velocity are Gaussian, and have covariance

$$\langle \boldsymbol{v}(\boldsymbol{r}, t)\, \boldsymbol{v}(\boldsymbol{r}', t) \rangle = \boldsymbol{\mathcal{P}}_{\boldsymbol{r},\boldsymbol{r}'},$$

where the projection operator is defined via its action on $\boldsymbol{w}(\boldsymbol{r})$

$$(\boldsymbol{\mathcal{P}} \boldsymbol{w})(\boldsymbol{r}) \equiv \int \boldsymbol{\mathcal{P}}_{\boldsymbol{r},\boldsymbol{r}'} \boldsymbol{w}(\boldsymbol{r}')\, d\boldsymbol{r}'.$$

Let us couple the fluctuating NS equation to a stochastic advection-diffusion for the concentration or density $c(\boldsymbol{r}, t)$ of a large collection of non-interacting *passive tracers*. For illustration purposes we can take a separable quadratic Hamiltonian (i.e., independent Gaussian fluctuations in velocity and concentration),

$$H(\boldsymbol{v},\, c) = H_{\boldsymbol{v}}(\boldsymbol{v}) + H_c(c) = \frac{\rho}{2} \int v^2\, d\boldsymbol{r} + \frac{k_B T}{2\epsilon} \int c^2\, d\boldsymbol{r},$$

and write the the model additive-noise tracer equation

$$\partial_t c = -\boldsymbol{v} \cdot \boldsymbol{\nabla} c + \chi \boldsymbol{\nabla}^2 c + \boldsymbol{\nabla} \cdot \left[(2\epsilon\chi)^{\frac{1}{2}}\, \boldsymbol{\mathcal{Z}}_c \right]. \tag{15}$$

Note that (15) is a conservation law because $\boldsymbol{v} \cdot \boldsymbol{\nabla} c = \boldsymbol{\nabla} \cdot (c\boldsymbol{v})$ due to incompressibility.

The coupled velocity-concentration system (14,15) can formally be written in the form (1). The chemical potential $\mu(c) = \partial H / \partial c \sim c$. The mobility operator can be written as a sum of a skew-adjoint and a self-adjoint part,

$$\boldsymbol{N} = \boldsymbol{M} - \boldsymbol{S} = -\left[\begin{array}{cc} \rho^{-1}\nu\left(\boldsymbol{\mathcal{P}}\boldsymbol{\nabla}^{2}\boldsymbol{\mathcal{P}}\right) & \boldsymbol{0} \\ \boldsymbol{0} & \epsilon\left(k_{B}T\right)^{-1}\left(\chi\boldsymbol{\nabla}^{2}\right) \end{array}\right], \quad (16)$$

$$-\rho^{-1}\left[\begin{array}{cc} \left(\boldsymbol{\mathcal{P}}\boldsymbol{\omega}\boldsymbol{\mathcal{P}}\right) & \boldsymbol{\mathcal{P}}\boldsymbol{\nabla}c \\ -\left(\boldsymbol{\nabla}c\right)^{T}\boldsymbol{\mathcal{P}} & 0 \end{array}\right] \quad (17)$$

where $\boldsymbol{\omega}$ is the antisymmetric vorticity tensor, $\boldsymbol{\omega}_{jk} = \partial v_k/\partial r_j - \partial v_j/\partial r_k$, and we used the vector identity

$$\boldsymbol{\omega}\boldsymbol{v} = -\left(\boldsymbol{\nabla}\times\boldsymbol{v}\right)\times\boldsymbol{v} = -\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\left(\frac{v^{2}}{2}\right).$$

Even though by skew symmetry the top right sub-block of $\boldsymbol{S}$ is nonzero, there is no coupling of concentration back in the velocity equation because

$$\left(\frac{\partial H}{\partial c}\right)\boldsymbol{\nabla}c = \left(\frac{dH_{c}}{dc}\right)\boldsymbol{\nabla}c = \boldsymbol{\nabla}H_{c}$$

is a gradient of a scalar and is eliminated by the projection. The velocity equation therefore remains of the form (14).

References

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