Now consider using Lax-Wendroff with diffusion

\[ u_t + a u_x = d u_{xxx} \]

First, figure out what the correct second-order Taylor series is:

\[ u_{tt} = -a (u_t)_x + d (u_t)_{xx} = \]
\[ = -a (-au_x + d u_{xx})_x + d (-au_x + d u_{xx})_{xx} \]

\[ u_{tt} = a^2 u_{xx} - 2ad u_{xxx} + d^2 u_{xxxxx} \]

\[ \Rightarrow u^{n+1} = u^n + (-au_x + d u_{xx}) \Delta t + \frac{\Delta t^2}{2} (a^2 u_{xx} - 2ad u_{xxx} + d^2 u_{xxxxx}) + \ldots \]
It is not a good idea to now proceed to discretize everything using centered differences. Instead, use Lax–Wendroff only for advection and Crank–Nicolson (implicit midpoint) for diffusion. How?

First, let us consider a more general splitting framework:

\[ U_t = AU + BU \]

Linear operators

\[
\begin{align*}
A &= -a \partial_x \\
B &= d U_{xx}
\end{align*}
\]

In our case
\[ U_t = (A+B)U \implies U^{n+1} = \left\{ I + (A+B)\Delta t + \frac{1}{2}(A^2 + B + AB + BA)\Delta t^2 \right\} U^n + O(\Delta t^3) \]

In our case, \( A^2 = a^2 2xx \), \( B^2 = d^2 u_{xxxx} \), \( AB = -ad u_{xxxx} = BA \)

Note, however, that generally \( AB \neq BA \) (non-constant coefficients, boundary conditions) so we should not assume this if we want a general approach.
So what Lax–Wendroff is doing is really
\[ u^{n+1} = \left\{ I + A \Delta t + \frac{1}{2} \tilde{A}^2 \Delta t^2 \right\} u^n \]
where \( \tilde{A}^2 \approx A^2 \) but not equal.

This is why it is not an MOL scheme—MOL would only have one \( A \)!

But for purposes of error analysis (second-order accuracy) we can treat Lax–Wendroff as doing \( \tilde{A}^2 = A^2 \).
First try:

\[ U^{n+1} - U^n = \frac{\Delta t}{t} \text{ Lax-Wendroff for advection} + \frac{d}{2} \left( \frac{U_{xx}^{n+1} + U_{xx}^n}{2} \right) \]

\[ \Downarrow \]

\[ \Rightarrow \quad \frac{U^{n+1} - U^n}{\Delta t} = (A + A^2 \Delta t) U^n + \frac{B}{2} (U^{n+1} + U^n) \]

\[ \Rightarrow \quad U^{n+1} = (I - \frac{B}{2} \Delta t)^{-1} \left\{ (A + A^2 \Delta t) U^n + (I + \frac{B}{2} \Delta t) U^n \right\} \]

\[ = (I - \frac{B}{2} \Delta t)^{-1} \left\{ U^n + \frac{B}{2} \Delta t + A \Delta t + \frac{A^2}{2} \Delta t^2 \right\} U^n \]
Expand
\[(I - \frac{B \Delta t}{2})^{-1} = I + \frac{B \Delta t}{2} + \frac{B^2 \Delta t^2}{4}\]
to get
\[u^{n+1} = \left( I + \frac{B \Delta t}{2} + \frac{B^2 \Delta t^2}{4} \right)\left( I + \frac{B \Delta t}{2} + A \Delta t + \frac{A^2 \Delta t^2}{2} \right) u^n\]
\[= \left( I + B \Delta t + A \Delta t + \frac{A^2}{2} \Delta t^2 + \frac{B^2 \Delta t^2}{2} + \frac{BA}{2} \Delta t^2 + O(\Delta t^3) \right) u^n\]

We are missing \(\frac{A B}{2} \Delta t^2\) ! NOT SECOND ORDER
How to fix this?

There are many ways up to second order:

\{ - Time splitting (e.g. Strang)
- Predictor-corrector schemes
\}

They are expensive because they require either two CN-solves or two Lax-Wendroff steps per time step.

Instead:

\[ \text{(LW)} \]

1. Solve \( U_t = AU_n + B U_n \) (source term)
2. Solve \( U_t = B U_n + U_t^{(LW)} \)

where the approximation of the Lax-Wendroff is now a source term.
Lax–Wendroff with spatial source term  \( n \)

\[ u_t + a u_x = S(x) \]

\( \not\in \) a function of time

\[ u_{tt} = -a (u_t)_x = -a (-a u_x + S)_x \]

\[ = a^2 u_{xx} - a S_x \]

So,

\[ u_i^{n+1} = u_i^n - \frac{a \Delta t}{2 \Delta x} (u_{i+1}^n - u_{i-1}^n) + S_i \Delta t \]

\[ + \frac{a^2 \Delta t^2}{2 \Delta x^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \]

\[ - \frac{a \Delta t^2}{2 \Delta x} (S_{i+1}^n - S_{i-1}^n) \]

( for example)
In our case
\[ S = B u^n = d u^n \]
\[ S_i^n = d \left( \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \right) \]

Algebraically, what we have done is
\[ u^{n+1} = (I + A \Delta t + \frac{A^2}{2} \Delta t^2 + \frac{AB}{2} \Delta t^2)u^n + B \Delta t \]

which is almost second-order accurate (now missing \( \frac{BA}{2} \Delta t^2 \) term)

But it would not be A-stable because diffusion is treated explicitly
Instead of Crank-Nicolson for diffusion but treat the Lax-Wendroff update as a source term:

\[
\frac{U^{n+1} - U^n}{\Delta t} = \frac{B}{2} (U^{n+1} + U^n) + \left( A \Delta t + \frac{A^2 \Delta t^2}{2} + \frac{AB \Delta t^2}{2} \right) U^n
\]

So now

\[
U^{n+1} = \left( I + \frac{BA \Delta t}{2} + \frac{B^2 \Delta t^2}{4} \right) U^n + \mathcal{O}(\Delta t^3) \leq \text{second order!}
\]

\[
\times \left( I + \frac{B \Delta t}{2} + A \Delta t + \frac{A^2 \Delta t^2}{2} + \frac{AB \Delta t^2}{2} \right) U^n
\]

\[
= \left[ I + (A + B) \Delta t + \frac{1}{2} \left( \frac{A^2}{2} + B^2 + AB + BA \right) \Delta t^2 \right] U^n
\]
How about stability? Is the only limitation now \( \frac{a^2 t}{4x} \leq C \approx 1 \)?

It turns out no, to get true stability for any diffusive CFL number we need some upwinding. So, instead of Lax-Wendroff consider Fromm's scheme. (with diffusion)

Fromm's scheme.

We need source term:

\[
U_t = -a u_x + s, \quad a > 0
\]

to Fromm's method.
Extrapolate state to faces at midpoint as we did before:

$$u_{j+1/2}^{n+1} = u_j^n + \frac{\Delta x}{2} (u_x)_j^n + \frac{\Delta t}{2} (u_t)_j^n$$

$$= u_j^n + \frac{\Delta x}{2} (u_x)_j^n + \frac{\Delta t}{2} (-a(u_x)_j^n + S_j)$$

$$= u_j^n + \frac{1}{2} (\Delta x - a\Delta t)(u_x)_j^n \text{ as before} + \frac{\Delta t}{2} S_j^n \text{ new term}$$

And here, $$S = dU_{xx}$$, so

$$S_j^n = \frac{d}{\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$
Now
\[ u_j^{n+1} = u_j^n - \Delta t \alpha (u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}) + S_j \cdot \Delta t \]
is Fromm's scheme with a source.

The extra term added in
\[ u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2} \]
\[ - \frac{a d \Delta t^2}{2 \Delta x^2} \left[ (u_{j+1} - 2u_j + u_{j-1}) - (u_j - 2u_{j-1} + u_{j-2}) \right] \]
\[ = - \frac{a d \Delta t^2}{2 \Delta x^2} \left[ u_{j+1} - 3u_j + 3u_{j-1} - u_{j-2} \right] \]
\[ \rightarrow -\frac{a d}{2} (u_{xxx})_j \Delta t^2 \]
Taylor series shows this is a discretization of \( \frac{\Delta B \Delta t^2}{2} = \frac{ad}{2} (U_{xxx})_j \Delta t^2 \)

but here \( U_{xxx} \) is upwind biased.

Compare this to what Lax-Wendrofft does, giving \( \frac{\Delta B \Delta t^2}{2} \) in the form

\[
-\frac{a d \Delta t^2}{2 \Delta x^3} \left[ (u_{j+2} - 2u_{j+1} + u_j) - (u_j - 2u_{j-1} + u_{j-2}) \right]
\]

\[
= -\frac{a d \Delta t^2}{2 \Delta x^3} \left[ u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2} \right]
\]

\[
\rightarrow -\frac{ad}{2} (U_{xxx})_j \Delta t^2
\]

But this is centered now.