

④ However, MOL is not the only approach. Also, it can be misleading to consider space and time as disjoint dimensions, and it is better to analyze the spatio-temporal error.

Let us first consider an example of a non-MOL scheme, the famous Lax-Wendroff scheme.

Although not originally developed this way, this is our first example of a semi-Lagrangian scheme.

⑤ So far we have considered Eulerian schemes: the grid is fixed and the fluid moves relative to the grid. But for pure advection it is much more natural to ~~use~~ follow the characteristics of the equation, which is a Lagrangian method:

$$u_t + \underline{a} \cdot \nabla u = 0, \quad \underline{a}(r, t) \text{ known}$$

Define a Lagrangian tracer:

$$\frac{d\xi(t)}{dt} = \underline{a}[\xi(t), t]$$

$$\Rightarrow \frac{du}{dt}[\xi(t), t] = 0 \Rightarrow u[\xi(t), t] = \text{const}$$

⑥ So we can ~~can~~ solve the pure advection equation by following the characteristics $\xi(t)$ along a collection of Lagrangian points

$$u[\xi(t), t] = u[\xi(0), 0]$$

This has many practical issues (discuss on board).

An alternative is to combine Eulerian and Lagrangian approaches. In Semi-Lagrangian approaches we read the values of $u(\tau, t)$ on a fixed

⑦ Eulerian grid, but we do this by following the characteristics backward in time to find the value from the past:

$$\left\{ \begin{array}{l} u(x_j, t_{n+1}) = u(\xi(t_n), t_n) \\ \frac{d\xi}{dt} = a(\xi, t) \quad (\text{Backward}) \\ \xi(t_{n+1}) = x_j \end{array} \right. \quad \text{from } t_{n+1} \rightarrow t_n$$

How do we now discretize this?

Let's focus on Lax-Wendroff

⑧

$$u_t + a u_x = 0$$

$$\Rightarrow u(x_j, t_{n+1}) = u(x_j - \bar{v}a, t_n)$$

Denote $\boxed{\nu = \frac{\bar{v}a}{h}}$ \rightarrow advective Courant or CFL number

$$\Rightarrow \bar{v}a = \nu h$$

Lax-Wendroff can be obtained by using quadratic interpolation to get $u(x_j - \bar{v}a, t_n)$:

Assume

$$\boxed{-1 \leq \nu \leq 1}$$

CFL condition

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Interpolate

$$u(x_j - \nu h) \approx \frac{1}{2} \nu(\nu+1) u(x_{j-1}) + (1-\nu^2) u(x_j) + \frac{1}{2} \nu(\nu-1) u(x_{j+1})$$

$$a \geq 0 \quad \underline{\text{or}} \quad a \leq 0$$

This leads to the scheme: Lax-Wendroff

$$w_j^{n+1} = w_j^n + \underbrace{\frac{a \bar{\tau}}{2h} (w_{j-1}^n - w_{j+1}^n)}_{\text{forward Euler for centered advection}} + \frac{1}{2} \left(\frac{a \bar{\tau}}{h} \right)^2 \underbrace{\left[w_{j-1}^n - 2w_j^n + w_{j+1}^n \right]}_{\text{second-order diffusive correction}}$$

Lax-Wendroff (the usual one) solves (8)

$$u_t = A u$$

$$u^{n+1} = \left\{ \underbrace{I + A \Delta t}_{\text{centered difference}} + \underbrace{\frac{1}{2} A^2 \Delta t^2}_{\text{centered diffusion} \neq (\text{centered diff})^2} \right\} u^n$$

$$\left(\frac{A^2 u}{2} \right)_i^n = \frac{a^2}{2} \frac{1}{2\Delta x} \left[\frac{u_{i+1+1} - u_{i+1-1}}{2\Delta x} - \frac{u_{i-1+1} - u_{i-1-1}}{2\Delta x} \right]$$

$$= \frac{a^2}{8\Delta x^2} [u_{i+2} - 2u_i + u_{i-2}]$$

$$\neq \frac{a^2}{2\Delta x^2} [u_{i+1} - 2u_i + u_{i-1}]$$

So what Lax-Wendroff is doing (9)
is really

$$u^{n+1} = \left\{ I + A \Delta t + \frac{1}{2} \tilde{A}^2 \Delta t^2 \right\} u^n$$

where $\tilde{A}^2 \approx A^2$ but not equal

This is why it is NOT an MOL
scheme - MOL would only have one A!

But for purposes of error analysis
(second-order accuracy) we can treat
Lax-Wendroff as doing $\tilde{A}^2 = A^2$

(10) The Lax-Wendroff scheme is not a MOL scheme in the usual sense. Note that it can be generalized to non-constant coefficients and higher dimensions (more to follow)

In either case (MOL or space-time), one-step difference schemes for linear PDEs look like

$$\boxed{B_0 w_{n+1} = B_1 w_n + \frac{G(t_n, t_{n+1})}{h}}$$

$\mathbb{R}^{m \times m}$ matrices
implicit part explicit

(11)

Global error
(space-time)

$$e_n = U_h(t_n) - w_n$$

Truncation error S_n

$$B_0 U_h(t_{n+1}) = B_1 U_h(t_n) + G(t_n, t_{n+1}) + \tau S_n$$

$$\Rightarrow B_0 e_{n+1} = B_1 e_n + \tau S_n$$

$$\Rightarrow \boxed{e_{n+1} = B_0^{-1} B_1 e_n + \delta_n} \quad \text{error growth}$$

$$\boxed{\delta_n = \tau B_0^{-1} S_n} \rightarrow \text{Local error (space-time)}$$

$$\text{If } B_0 = I \Rightarrow \delta_n = S_n$$

$$\textcircled{12} \begin{cases} E_{n+1} = B E_n + \delta_n \\ B = B_0^{-1} B_1 \end{cases}$$

Consistency: $\|g_n\| \rightarrow 0$ as $(\tau, h) \rightarrow 0$

Stability: $\begin{cases} \|B^n\| \leq K_n, n \geq 0, n \leq \frac{T}{\tau} \\ K_n \text{ independent of } \tau, h \end{cases}$

If we assume the natural

$$\|B_0^{-1}\| \leq C \quad \Rightarrow \quad \text{independent of } h, \tau$$

$$\|\delta_n\| \leq C \tau \|g_n\| \quad \text{so}$$

so consistency bounds the local error

(13)

Consistency + stability \Leftrightarrow convergence

(Lax Equivalence theorem)

$$\|E_n\| \leq K \|E_0\| + K \sum_{k=0}^{n-1} \|\delta_k\|$$

$$\|E_n\| \leq K \|E_0\| + K C t_n \max_{0 \leq h \leq n-1} \|S_h\|$$

Error estimate / bound

The Lax-Wendroff scheme is consistent and stable if $|v| \leq 1$, which is a typical stability condition for a direction.

Space - TIME METHODS

(1)

FOR ADVECTION - DIFFUSION

CFD FALL 2014
A. DONEV

(some notes & comments)

Recall that Lax-Wendroff is NOT
an MOL (method-of-lines) scheme,
it is a space-time scheme.

It's local truncation error is:

$$\mathcal{J} = -\frac{1}{6} a \Delta x^2 (1 - \nu^2) u_{xxx} + O(\Delta t^3)$$

where $\nu = \frac{a \Delta t}{\Delta x}$ is CFL number

$$S = -\frac{1}{6} u_{xxxx} (\Delta x^2 - a^2 \Delta t^2) \quad (2)$$

which is a sum of a spatial and a temporal error just like MOL schemes. But this is NOT always the case.

Consider the Lax-Friedrichs method

$$u_i^{n+1} = \frac{1}{2} (u_{i-1}^n + u_{i+1}^n) - (\Delta t) a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

Stabilization
(not MOL)

Centered Difference
+ Euler (UNSTABLE)

Stable if $CFL \leq 1$

Doing Taylor series on

(3)

$$\frac{1}{2} (u_{i-1}^n + u_{i+1}^n) = u_i^n + \frac{1}{2} \Delta x^2 u_{xx}$$

we see that the modified equation is

$$u_t = -a u_x + \underbrace{\frac{1}{2} \frac{\Delta x^2}{\Delta t} u_{xx}}_{\text{Numerical (artificial) dissipation}}$$

The Lax-Friedrichs method is not even consistent if $\Delta t \rightarrow 0$

keeping Δx fixed: One must refine both space and time together for space-time schemes

The error for Lax-Friedrichs is (4)

$$g = \frac{1}{2} a \Delta x (v^{-1} - v) u_{xx} + O(\Delta t^2)$$

So it must be run with $v \approx 1$

When talking about non-MOL schemes we are looking at space-time error

$$v = \text{const} \Rightarrow \Delta t = O(\Delta x)$$

(for advection problems)

Note that for explicit (conditionally stable) methods we can never refine in space only due to $v \leq 1$ limit

① HIGH-RESOLUTION ADVECTION

A. DONER, CFD, Fall 2014

RECALL THE LAX-WENDROFF SCHEME FOR

$$u_t + a u_x = 0$$

$$u_i^{n+1} = b_{-1} u_{i-1}^n + b_0 u_i^n + b_1 u_{i+1}^n$$

$$\left\{ \begin{array}{l} b_{-1} = \frac{1}{2} c (1+c) \\ b_0 = 1 - c^2 \\ b_1 = -\frac{1}{2} c (1-c) \end{array} \right.$$

where $c = \frac{a \Delta t}{\Delta x}$

which we derived as a semi-Lagrangian
scheme by tracing characteristics backward

(2)

Note that the coefficients b_{-1}, b_0, b_1 can easily be derived by performing truncation error (in both space and time) analysis to obtain the 2nd order accuracy conditions:

$$\begin{cases} b_{-1} + b_0 + b_1 = 1 \\ b_{-1} + b_1 = c^2 \\ b_{-1} - b_1 = -c \end{cases}$$

But here alternative generalized our focus will be on derivations that could be to other equations

③ ANOTHER WAY TO DERIVE LAX-WENDROFF:

$$u_t + a u_x = 0 \Rightarrow$$

$$u_{tt} = (u_t)_t = (-a u_x)_t = (-a u_t)_x \\ = + a^2 (u_x)_x = a^2 u_{xx}$$

$$u(\Delta t) = u(0) + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} =$$

$$= u(0) - \Delta t u_x + \frac{\Delta t^2 a^2}{2} u_{xx}$$

Use centered
advection

Use centered
diffusion

This kind of game swapping time
for space derivatives is common!

13 We can also try a midpoint scheme where we try to extrapolate state to the faces at the midpoint:

$$\begin{aligned} \left(f_{\bar{i}+1/2} \right)^n &\approx a u_{\bar{i}+1/2}^{n+1/2} \\ &\approx a u \left(x_{\bar{i}} + \frac{\Delta x}{2}, t^n + \frac{\Delta t}{2} \right) \end{aligned}$$

Use Taylor series and the PDE
 $u_t = -f_x = -a u_x$

$$\begin{aligned} u_{\bar{i}+1/2}^{n+1/2} &\approx u_i^n + \frac{\Delta t}{2} (u_t)_i^n + \frac{\Delta x}{2} (u_x)_i^n \\ &= u_i^n - \frac{a \Delta t}{2} (u_x)_i^n + \frac{\Delta x}{2} (u_x)_i^n \end{aligned}$$

(14)

$$u_{\bar{i}+1/2}^{n+1/2} \approx u_i^n + \frac{1}{2} (\Delta x - a \Delta t) (u_x)_i^n$$

Now we need to approximate the slope by finite differences

Use centered approximation

$$(u_x)_i^n \approx \frac{u_{\bar{i}+1}^n - u_{\bar{i}-1}^n}{2\Delta x}$$

$$f_{\bar{i}+1/2}^n = a u_i^n + \frac{a(1-c)}{4} (u_{\bar{i}+1}^n - u_{\bar{i}-1}^n)$$

which is the same as Fromm's method but derived from a different perspective