Incompressible Flow

CFD Spring 2013

\[ \nabla_t + \nabla \Pi = - (\nabla \cdot \nabla) \nabla + \nu \nabla^2 \nabla + \text{other} \]

Lagrange multiplier

Advection

Kinematic viscosity

\[ \nabla \nabla \nabla = 0 \quad \text{(incompressibility)} \]

Note \[ \nabla \cdot \nabla \nabla = \nabla \cdot \nabla (\nabla \nabla) \]

\[ u_i \rightarrow u_j \partial_j u_i \equiv \partial_j (u_i u_j) \]

Since \[ u_j \partial_j u_i + u_i \partial_i u_j \]
\[ \begin{align*}
\partial_t u_i + \partial_j P &= - \nu_j \partial_j u_i + \nu \partial_j^2 u_i \\
\partial_j u_j &= 0, \quad i = 1, 2, \ldots, d \quad (d = 2 \quad or) \\
\text{Implied summation (Einstein) convention for repeated indices}
\end{align*} \]

This form of the equations applies only if density is constant

\[ S = \text{const.} \]

\[ \nu = \frac{\eta}{S} \quad \text{viscosity} \]
Otherwise, one needs to solve

\[
\begin{aligned}
(\mathbf{g} \mathbf{u})_t + \nabla \cdot p &= -\nabla \cdot (\mathbf{g} \mathbf{u} \otimes \mathbf{u}) + \nabla \cdot (\mathbf{g} \mathbf{u}) \\
\text{momentum}
\end{aligned}
\]

\text{conservation}

\[
\begin{aligned}
\mathbf{S}_{t} + \mathbf{u} \cdot \nabla \mathbf{S} &= 0 \quad \text{continuity equation (conservation of mass)} \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
\]

\text{Equivalent formulation:}

\[
\begin{aligned}
\mathbf{S}_{t} + \mathbf{g} \mathbf{u} \cdot \nabla \mathbf{u} &= \nabla \cdot (\mathbf{g} \mathbf{u}) + \mathbf{S}_{\mathbf{g}} \\
\text{advection}
\end{aligned}
\]

\text{stress tensor}
Here the stress-tensor

\[ \sigma = -p \mathbf{I} + \eta (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \]

mechanical stress
dissipative stress (tensor)

\[ \sigma_{ij} = -p \delta_{ij} + \eta (\partial_i u_j + \partial_j u_i) \]

Note that if \( \eta = \text{const} \)

\[ \partial_j \sigma_{ji} = -\partial_i p + \eta (\partial_j^2 u_i + \partial_j \partial_i u_j) = \]

\[ = -\partial_i p + \eta \partial_j^2 u_i + \eta \partial_j \partial_i u_j = 0 \]

\[ \Rightarrow \partial_i \sigma_{ij} = -\partial_j p + \eta \partial_j^2 u_i + \eta \partial_j \partial_i u_j = 0 \]

\[ \Rightarrow \sigma_{ij} = \text{const} \]
To summarize variable-density variable-viscosity equations:

\[
\begin{align*}
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\mathbf{\nabla} p + \mathbf{\nabla} \cdot \left[ \eta \left( \mathbf{\nabla} \mathbf{u} + (\mathbf{\nabla} \mathbf{u})^T \right) \right] + \text{other} \\
\mathbf{S}_t + \mathbf{u} \cdot \nabla \mathbf{S} &= 0 \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

But for now let's focus on the constant-coefficient case

\[ \eta = \text{const} \quad \gamma = \text{const} \quad \gamma = \frac{n}{\eta} \]
\[
\begin{aligned}
\n\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \gamma \nabla^2 u + \text{forcing} + \gamma \\
\n\n\n\text{Interface conditions:} \\
\n\n\n\n\n\n\n\text{Concentration or density of a \underline{passively} advected scalar (e.g., a pollutant advected by the flow of air).} \\
\n\text{As we can see, these are basically advection-diffusion equations with a twist:} \\
\n\n\n\n\n\text{\rightarrow } u \cdot \nabla u \text{ is non-linear} \\
\text{\rightarrow The equations are \underline{constrained} } \nabla \cdot u = 0 \\
\text{\rightarrow Pressure has no evolution law}
\end{aligned}
\]
Formally, the NS equations are a differential-algebraic system of equations (DAE) of index 2.

Even if they were simple ODEs, they would be non-trivial to integrate in time.

It is possible to formally eliminate the pressure to get the pressure-free formulation:

\[ u_t = P \left[ -u \cdot \nabla u + \nabla^2 u + f \right] \]

where \( P \) is an integro-differential projection operator.
Hodge Decomposition (or) Helmholtz Theorem

Let \( \mathbf{v} \) be a vector field on a bounded domain in \( \mathbb{R}^3 \), smooth.

\[
\mathbf{v} = -\nabla \phi + \nabla \times \mathbf{A} \quad \text{uniquely}
\]

irrotational part
divergence free part

\[
\mathbf{v} = -\nabla \phi + \mathbf{u} \quad , \quad \nabla \cdot \mathbf{u} = 0
\]

If \( \mathbf{v} \) decays at infinity or vanishes on boundary of domain, one can write explicitly.
where $P$ is a projection operator that takes a vector field and projects it onto the space of divergence-free vector fields.

\[ u = Pu \]

$L_2$ projection onto $\nabla \cdot u = 0$

\[
P u = \frac{1}{4\pi} \mathcal{D} \times \int \frac{\mathcal{D}' \times u(r')}{|r-r'|} \, dV'
\]

\[
= u + \frac{1}{4\pi} \mathcal{D} \int \frac{\mathcal{D}' \cdot u(r')}{|r-r'|} \, dV'
\]
\[
\begin{align*}
\Psi(r) &= \frac{1}{4\pi} \int \frac{V(r')}{|r-r'|} \, dv' \\
\vec{A}(r) &= \frac{1}{4\pi} \int \frac{\vec{V} \times V(r')}{|r-r'|} \, dv'
\end{align*}
\]

Note that
\[\nabla \cdot V = -\nabla^2 \Psi\]

Poisson equation for \(\Psi\)

and \(-\frac{1}{4\pi \pi} \frac{1}{|r-r'|}\) is the Green's function for this Poisson equation

\[U = V + \nabla \Psi = V - \nabla (\nabla^2) \Psi \]
defines \(\mathcal{P} \Psi\)
Boundary Conditions

At a physical boundary, the following BCs are allowed: (4 types)

1. Normal component
   - normal velocity
     1. \( \vec{u} \cdot \vec{n} = u_n \)

2. Normal stress (traction)
   \[ \vec{n} \cdot \sigma \cdot \vec{n} = -p + 2\eta \frac{\partial}{\partial n} (\vec{n} \cdot \vec{n}) \]

3. Tangential component
   - tangential stress
   \[ \sigma \cdot \vec{t} \cdot \vec{n} = \eta \left[ \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{u} \cdot \vec{n}}{\partial n} \right] \]

Note: Periodic boundaries are not real (physical) boundaries!

Planar boundaries specified
So the four options are:

A \{ \begin{align*} \vec{u} \cdot \vec{n} &= u_n \\ \vec{u}_t & \text{ specified (i.e. Dirichlet for } u) \end{align*} \}

B \{ \begin{align*} \vec{u} \cdot \vec{n} &= u_n \\ \eta \left[ \frac{\partial \vec{u}_t}{\partial n} + \frac{\partial \vec{u}_n}{\partial \vec{r}} \right] &= \vec{f}_t \end{align*} \}

C \{ \begin{align*} -p + 2\eta \frac{\partial u_n}{\partial n} &= \vec{f}_n \\ \vec{u}_t & \end{align*} \}

D \{ \begin{align*} -p + 2\eta \frac{\partial u_n}{\partial n} &= \vec{f}_n \\ \eta \left[ \frac{\partial \vec{u}_t}{\partial n} + \frac{\partial \vec{u}_n}{\partial \vec{r}} \right] &= \vec{f}_t \end{align*} \}

normal velocity
normal stress
normal stress
Dirichlet for stress
tangential stress (slip BCs)
tangential stress
normal and tangential
(NO-SLIP BCs)
In practice one often wants "outflow" or transparent BCs but these are not proper physical BCs since usually the physical conditions are unknown (artificial boundaries).

Note: For non-ideal flow, Euler equations, one can only specify normal component: either normal velocity or pressure. Boundary layers will occur when viscosity is weak (recall cell Pélet number).