INCOMPRESSIBLE FLOW

CFD SPRING 2013

$$\begin{cases} \vec{u}_t + \underbrace{\nabla \Pi}_{\text{Lagrange multiplier for}} = -\underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{advection}} + \nu \nabla^2 \vec{u} + \text{other} \\ \nabla\cdot\vec{u} = 0 \quad \text{(incompressibility)} \end{cases}$$

ν kinematic viscosity

Note $\quad u\cdot\nabla u \equiv \nabla\cdot(u\otimes u)$

$$u_i \longrightarrow u_j \partial_j u_i \equiv \partial_j (u_i u_j)$$

$$\text{since} \quad \partial_j (u_i u_j) = u_j \partial_j u_i + u_i \underbrace{\cancel{\partial_j u_j}}_{\text{zero}}$$

$$\begin{cases} \partial_t u_i + \partial_i p = -u_j \partial_j u_i + \nu \partial_j \partial_j u_i \\ \partial_j u_j = 0 \qquad , i = 1, 2, \dots d \quad \left(\begin{smallmatrix} d=2 \text{ or} \\ d=3 \end{smallmatrix}\right) \end{cases}$$

Implied summation (Einstein) convention for repeated indices

This form of the equations applies only if density is constant

$$\rho = \text{const.}$$

$$\nu = \frac{\eta}{\rho} \leftarrow \text{viscosity}$$

Otherwise, one needs to solve (3)

$$\begin{cases} (\rho \vec{u})_t + \nabla p = -\nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla \cdot (\overleftrightarrow{\sigma_v}) \\ \quad + \text{ others like } + \rho \vec{g} \text{ gravity} \end{cases}$$

momentum conservation

$$\boxed{\begin{array}{l} \rho_t + u \cdot \nabla \rho = 0 \leftarrow \text{continuity equation (conservation of mass)} \\ \nabla \cdot u = 0 \\ \text{Equivalent formulation:} \\ \rho u_t + \underbrace{\rho u \cdot \nabla u}_{\text{advection}} = \nabla \cdot (\underbrace{\sigma}_{\text{stress tensor}}) + \rho \vec{g} \end{array}}$$

Here the stress-tensor

$$\overleftrightarrow{\sigma} = \underbrace{-p \overleftrightarrow{I}}_{\text{mechanical stress}} + \underbrace{\eta \left(\vec{\nabla}\vec{u} + \vec{\nabla}^T \vec{u} \right)}_{\text{viscous stress (tensor)}}$$

$$\sigma_{ij} = -p\,\delta_{ij} + \eta \left(\partial_i u_j + \partial_j u_i \right)$$

Note that if $\eta = \text{const}$

$$\partial_j \sigma_{ji} = -\partial_i p + \eta \left(\partial_j^2 u_i + \partial_j \partial_i u_j \right) =$$

$$= -\partial_i p + \eta\, \partial_j^2 u_i + \eta\, \partial_i \underbrace{(\partial_j u_j)}_{\text{zero}}$$

$$\Rightarrow \boxed{\nabla \cdot \overleftrightarrow{\sigma} = -\nabla p + \eta\, \nabla^2 u}$$

To summarize <u>variable-density variable-viscosity</u> equations:

$$\begin{cases} u_t + u\cdot\nabla u = -\rho^{-1}\nabla p + \rho^{-1}\nabla\cdot\left[\eta\left(\nabla v + \nabla^T v\right)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad + \text{other} \\ \rho_t + u\cdot\nabla\rho = 0 \\ \nabla\cdot u = 0 \end{cases}$$

But for now let's focus on the constant-coefficient case

$$\rho = \text{const} \qquad \eta = \text{const}\ ; \quad \nu = \frac{\eta}{\rho}$$

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \nu \nabla^2 u + \text{forcing } f \\ \nabla \cdot u = 0 \\ c_t + u \cdot \nabla c = \chi \nabla^2 c + \text{forcing } g \end{cases}$$

Concentration or density of a <u>passively advected</u> scalar (e.g., a pollutant advected by the flow of air)

As we can see, these are basically <u>advection - diffusion</u> equations with a twist:

- → $u \cdot \nabla u$ is nonlinear
- → The equations are <u>CONSTRAINED</u> $\nabla \cdot u = 0$
- → Pressure has no evolution law

Formally, the NS equations are a differential-algebraic system of equations (DAE) of index 2.

Even if they were simple ODEs they would be non-trivial to integrate in time!

It is possible to formally eliminate the pressure to get the pressure-free formulation

$$\boxed{u_t = \mathbb{P}\left[-u\cdot\nabla u + \nu\nabla^2 u + f\right]}$$

where $\mathbb{P}$ is an integro-differential projection operator

NEXT

HODGE DECOMPOSITION (or) HELMHOLTZ THEOREM

Let v be a vector field on a bounded domain in R^3, smooth.

$$\vec{v} = \underbrace{-\nabla\varphi}_{\text{irrotational part}} + \underbrace{\nabla\times\vec{A}}_{\text{divergence free part}} \qquad \underline{\text{uniquely}}$$

$$v = -\nabla\varphi + u \quad , \quad \nabla\cdot u = 0$$

If v decays at infinity or vanishes on boundary of domain, one can write explicitly

where $\underline{P}$ is a projection operator 9
that takes a vector field and
projects it onto the space of
divergence-free vector fields

$$u = \underline{P}\, v$$

$\nwarrow$ L_2 projection onto $\nabla \cdot u = 0$

$$\left\{ \begin{aligned} \underline{P}\, v &= \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times v(r')}{|r-r'|}\, dV' \\ &= v + \frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot v(r')}{|r-r'|}\, dV' \end{aligned} \right.$$

$$\begin{cases} \varphi(r) = \dfrac{1}{4\pi} \displaystyle\int_V \dfrac{\nabla' \cdot \vartheta(r')}{|r-r'|} \, dV' \\[2ex] \vec{A}(r) = \dfrac{1}{4\pi} \displaystyle\int_V \dfrac{\nabla' \times \vartheta(r')}{|r-r'|} \, dV' \end{cases}$$

Note that

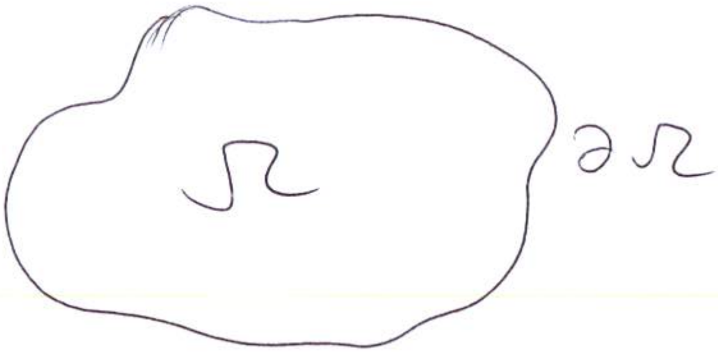
$\boxed{\nabla \cdot \vartheta = -\nabla^2 \varphi}$ Poisson equation for φ

and $-\dfrac{1}{4\pi} \dfrac{1}{|r-r'|}$ is the Green's function for this Poisson equation

$$u = \vartheta + \nabla \varphi = \vartheta - \nabla \left(\nabla^{-2} \right) \nabla \cdot \vartheta$$

$\underline{\text{defines}}$ $\equiv \underline{P}\, \vartheta$

Boundary Conditions



Note: Periodic boundaries are not real (physical) boundaries!

At a <u>physical boundary</u>, the following BCs are allowed: (4 types)

<u>normal component</u>	<u>tangential comp.</u>
<u>Specify</u>: normal velocity ① $\vec{u}\cdot\vec{n} = u_n$	① $\vec{u} - \vec{u}\cdot\vec{n} = \vec{u}_\tau$
normal stress (traction) ② $\vec{n}\cdot\overleftrightarrow{\sigma}\cdot\vec{n} = -p + 2\eta\frac{\partial}{\partial n}(\vec{n}\cdot\vec{n})$ specified	② $\vec{\tau}\cdot\overleftrightarrow{\sigma}\cdot\vec{n} = \eta\left[\frac{\partial \vec{u}}{\partial n}\tau + \frac{\partial u_n}{\partial \vec{\tau}}\right]$ tangential stress

$\boxed{\text{PLANAR BOUNDARIES}}$

So the four options are:

(A) $\begin{cases} \vec{u}\cdot\vec{n} = u_n \\ \vec{u}_{\tau} \text{ specified} \end{cases}$ (i.e. Dirichlet for u)
normal and tangential
(NO-SLIP BCs)

(B) $\begin{cases} \vec{u}\cdot\vec{n} = u_n \\ \eta\left[\dfrac{\partial \vec{u}_{\tau}}{\partial n} + \dfrac{\partial u_n}{\partial \vec{\tau}}\right] = \vec{f}_{\tau} \end{cases}$ normal velocity
tangential stress
(SLIP BCs)

(C) $\begin{cases} -p + 2\eta \dfrac{\partial}{\partial n} u_n = f_n \\ \vec{u}_{\tau} \end{cases}$ normal stress
tangential velocity

(D) $\begin{cases} -p + 2\eta \dfrac{\partial u_n}{\partial n} = f_n \\ \eta\left[\dfrac{\partial \vec{u}_{\tau}}{\partial n} + \dfrac{\partial u_n}{\partial \vec{\tau}}\right] = \vec{f}_{\tau} \end{cases}$ Dirichlet for stress

In practice one often wants "outflow" or transparent BCs but these are not proper physical BCs since usually the physical conditions are <ins>unknown</ins> (<ins>artificial boundaries</ins>).

<ins>Note</ins>: For <ins>inviscid flow</ins>, Euler equations, one can only specify: <ins>normal component</ins>: either normal velocity or pressure. <ins>Boundary layers</ins> will occur when viscosity is weak (recall cell Péclet number)