INCOMPRESSIBLE FLOW

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$$\begin{cases} \vec{u}_t + \underbrace{\nabla \Pi}_{\text{Lagrange multiplier for}} = \underbrace{-(\vec{u}\cdot\nabla)\vec{u}}_{\text{advection}} + \underset{\substack{\uparrow \\ \text{kinematic viscosity}}}{\nu} \nabla^2 \vec{u} + \text{other} \\ \nabla\cdot\vec{u} = 0 \quad \text{(incompressibility)} \end{cases}$$

Note $\quad u\cdot\nabla u \equiv \nabla\cdot(u\otimes u)$

$$u_i \longrightarrow u_j \partial_j u_i \equiv \partial_j (u_i u_j)$$

$$\text{since} \quad \partial_j (u_i u_j) = u_j \partial_j u_i + u_i \cancel{\partial_j u_j}^{\text{zero}}$$

$$\begin{cases} \partial_t u_i + \partial_i p = -u_j \partial_j u_i + \nu \partial_j \partial_j u_i \\ \partial_j u_j = 0 \qquad , i = 1, 2, \dots d \quad \left(\begin{smallmatrix} d=2 \text{ or} \\ d=3 \end{smallmatrix}\right) \end{cases}$$

Implied summation (Einstein) convention for <u>repeated</u> indices

This form of the equations applies only if <u>density is constant</u>

$$\rho = \text{const.}$$

$$\nu = \frac{\eta}{\rho} \leftarrow \text{viscosity}$$

Otherwise, one needs to solve

$$\left\{ \begin{array}{l} (\rho \vec{u})_t + \nabla p = -\nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla \cdot (\overleftrightarrow{\sigma_v}) \\ \qquad\qquad + \text{others like } + \rho \vec{g} \text{ gravity} \end{array} \right.$$

momentum conservation

$$\boxed{\begin{array}{l} \rho_t + u \cdot \nabla \rho = 0 \quad \leftarrow \text{ continuity equation (conservation of mass)} \\ \nabla \cdot u = 0 \\ \text{Equivalent formulation:} \\ \rho u_t + \underbrace{\rho u \cdot \nabla u}_{\text{advection}} = \nabla \cdot (\underbrace{\sigma}_{\text{stress tensor}}) + \rho \vec{g} \end{array}}$$

Here the <u>stress-tensor</u>

$$\overleftrightarrow{\sigma} = \underbrace{-p\,\overleftrightarrow{I}}_{\text{mechanical stress}} + \underbrace{\eta\left(\vec{\nabla}\vec{u} + \vec{\nabla}^T\vec{u}\right)}_{\text{viscous stress (tensor)}}$$

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\partial_i u_j + \partial_j u_i\right)$$

Note that if $\eta = \text{const}$

$$\partial_j \sigma_{ji} = -\partial_i p + \eta\left(\partial_j^2 u_i + \partial_j \partial_i u_j\right) =$$

$$= -\partial_i p + \eta\,\partial_j^2 u_i + \eta\,\partial_i \underbrace{\cancel{\left(\partial_j u_j\right)}}_{\text{zero}}$$

$$\Rightarrow \boxed{\nabla\cdot\overleftrightarrow{\sigma} = -\nabla p + \eta\,\nabla^2 u}$$

To summarize <u>variable-density</u> <u>variable-viscosity</u> equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\rho^{-1} \nabla p + \rho^{-1} \nabla \cdot \left[\eta \left(\nabla u + \nabla^T u \right) \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad + \text{other} \\ \rho_t + u \cdot \nabla \rho = 0 \\ \nabla \cdot u = 0 \end{cases}$$

But for now let's focus on the constant-coefficient case

$$\rho = \text{const} \qquad \eta = \text{const} \; ; \; \nu = \frac{\eta}{\rho}$$

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \nu \nabla^2 u + \text{forcing } f \\ \nabla \cdot u = 0 \\ c_t + u \cdot \nabla c = \chi \nabla^2 c + \text{forcing } g \end{cases}$$

Concentration or density of a <u>passively advected</u> scalar (e.g., a pollutant advected by the flow of air)

As we can see, these are basically <u>advection-diffusion</u> equations with a twist:

- → $u \cdot \nabla u$ is nonlinear
- → The equations are <u>CONSTRAINED</u> $\nabla \cdot u = 0$
- → Pressure has no evolution law

Formally, the NS equations are a <u>differential-algebraic system</u> of equations (DAE) of <u>index 2</u>.

Even if they were simple ODEs they would be <u>non-trivial</u> to integrate in time!

It is possible to formally eliminate the pressure to get the <u>pressure-free formulation</u>

$$\boxed{u_t = \mathbb{P}\left[-u\cdot\nabla u + \nu\nabla^2 u + f\right]}$$

where $\mathbb{P}$ is an integro-differential projection operator $\xrightarrow{\text{NEXT}}$

HODGE DECOMPOSITION (or) HELMHOLTZ THEOREM

Let v be a vector field on a bounded domain in R^3, smooth.

$$\vec{v} = \underbrace{-\nabla\varphi}_{\text{irrotational part}} + \underbrace{\nabla\times\vec{A}}_{\text{divergence free part}} \quad \underline{\text{uniquely}}$$

$$v = -\nabla\varphi + u \quad , \quad \nabla\cdot u = 0$$

If v decays at infinity or vanishes on boundary of domain, one can write explicitly

where $\underline{P}$ is a <u>projection operator</u> (9)

that takes a vector field and projects it onto the space of divergence-free vector fields

$$u = \underline{P}\, v$$

$\leftarrow L_2$ projection onto $\nabla \cdot u = 0$

$$\left\{ \begin{aligned} \underline{P}\, v &= \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times v(r')}{|r - r'|} \, dV' \\ &= v + \frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot v(r')}{|r - r'|} \, dV' \end{aligned} \right.$$

$$\begin{cases} \varphi(r) = \dfrac{1}{4\pi} \displaystyle\int_V \dfrac{\nabla' \cdot v(r')}{|r-r'|} dv' \\[2ex] \vec{A}(r) = \dfrac{1}{4\pi} \displaystyle\int_V \dfrac{\nabla' \times v(r')}{|r-r'|} dv' \end{cases}$$

Note that

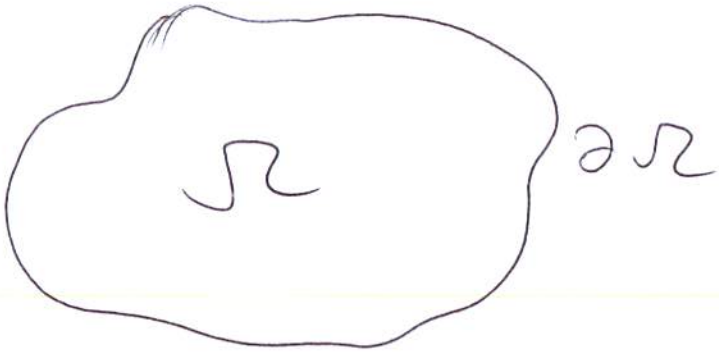
$\boxed{\nabla \cdot v = -\nabla^2 \varphi}$ Poisson equation for φ

and $-\frac{1}{4\pi}\frac{1}{|r-r'|}$ is the Green's function for this Poisson equation

$$u = v + \nabla\varphi = v - \nabla(\nabla^{-2})\nabla \cdot v$$

$$\underline{\text{defines}} \quad \equiv \underline{P}\, v$$

Boundary Conditions



Note: Periodic boundaries are not real (physical) boundaries!

At a <u>physical boundary</u>, the following BCs are allowed: (4 types)

<u>normal component</u>

<u>Specify</u>: normal velocity

(1) $\vec{u}\cdot\vec{n} = u_n$

normal stress (traction)

(2) $\vec{n}\cdot\overleftrightarrow{\sigma}\cdot\vec{n} = -p + 2\eta\frac{\partial}{\partial n}(\vec{n}\cdot\vec{n})$

$\boxed{\text{PLANAR BOUNDARIES}}$ specified

<u>tangential comp.</u>

(1) $\vec{u} - \vec{u}\cdot\vec{n} = \vec{u}_\tau$

(2) $\vec{\tau}\cdot\overleftrightarrow{\sigma}\cdot\vec{n} =$

$\eta\left[\frac{\partial \vec{u}_\tau}{\partial n} + \frac{\partial u_n}{\partial \vec{\tau}}\right]$

tangential stress

So the four options are:

(A) $\begin{cases} \vec{u}\cdot\vec{n} = u_n \\ \vec{u}_{\vec{\tau}} \text{ specified} \end{cases}$ (i.e. Dirichlet for u) normal <u>and</u> tangential (NO-SLIP BCs)

(B) $\begin{cases} \vec{u}\cdot\vec{n} = u_n \\ \eta\left[\dfrac{\partial \vec{u}_{\vec{\tau}}}{\partial n} + \dfrac{\partial u_n}{\partial \vec{\tau}}\right] = \vec{f}_{\vec{\tau}} \end{cases}$ normal velocity, tangential stress (SLIP BCs)

(C) $\begin{cases} -p + 2\eta \dfrac{\partial}{\partial n} u_n = f_n \\ \vec{u}_{\vec{\tau}} \end{cases}$ normal stress, tangential velocity

(D) $\begin{cases} -p + 2\eta \dfrac{\partial u_n}{\partial n} = f_n \\ \eta\left[\dfrac{\partial \vec{u}_{\vec{\tau}}}{\partial n} + \dfrac{\partial u_n}{\partial \vec{\tau}}\right] = \vec{f}_{\vec{\tau}} \end{cases}$ Dirichlet for stress

In practice one often wants "outflow" or transparent BCs but these are not proper physical BCs since usually the physical conditions are <u>unknown</u> (<u>artificial boundaries</u>).

<u>Note</u>: For <u>inviscid flow</u>, Euler equations, one can only specify: <u>normal component</u>: either normal velocity or pressure. <u>Boundary layers</u> will occur when viscosity is weak (recall cell Péclet number)

VORTICITY

$$\omega = \nabla \times u \quad \Rightarrow \quad \nabla \cdot \omega = 0$$

$\omega = 0 \rightarrow$ potential flow

$$u \cdot \nabla u = \frac{1}{2} \nabla |u|^2 - u \times (\overbrace{\nabla \times u}^{\omega})$$

(vector identity)

Note $\nabla \times (\nabla p) = 0$

$$\boxed{\begin{aligned} \partial_t u + \omega \times u + \nabla \tilde{p} &= \nu \nabla^2 u + f \\ \text{Vorticity formulation} \qquad &= -\nu \nabla \times \omega + f \end{aligned}}$$

where we used $\nabla \times (\nabla \times v) = \nabla (\nabla \cdot v) - \nabla^2 v$

Apply a curl to equation and use

$$\nabla \times (v \times w) = w \cdot \nabla v - v \cdot \nabla w$$

to get <u>vorticity equation</u>:

$$\boxed{\partial_t w + v \cdot \nabla w = w \cdot \nabla v + \nu \nabla^2 w}$$

In two dimensions, $\vec{v} = (u, v, 0)$

$w \cdot \nabla v = 0$, so there is no vorticity generation:

$$\partial_t w + v \cdot \nabla w = \nu \nabla^2 w \quad \text{in 2D}$$

Define a <u>stream function</u> ψ

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

i.e. $u = (\nabla \psi) \times \hat{z}$

to get Poisson equation for ψ

$$\boxed{\nabla^2 \psi = -\omega} \quad \text{in } \Omega \quad (2D)$$

$$u \xrightarrow[\nabla \times]{} \omega \xrightarrow[\nabla^{-2}]{} \psi \xrightarrow[\nabla \times \hat{z}]{} u$$

$$\partial_t \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega$$

$$u = -\nabla \left[(\nabla^{-2} \omega) \times \hat{z} \right]$$

Only vorticity appears!