HYPERBOLIC CONSERVATION LAWS

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LET'S GO BACK to

$$q_t + (f(q))_x = 0$$

FINITE - VOLUME METHOD

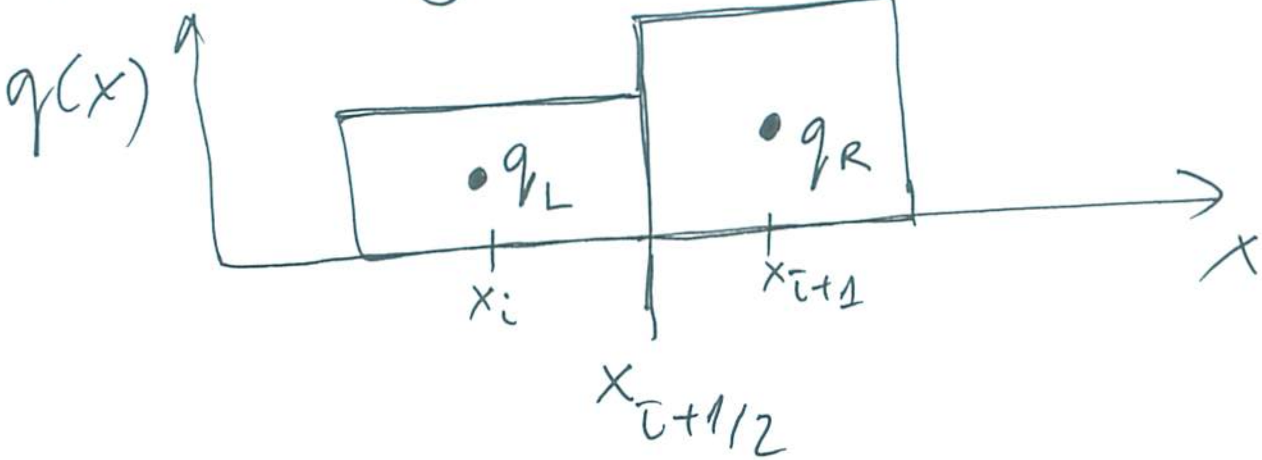
$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left[f_{i+1/2}^n - f_{i-1/2}^n \right]$$

where

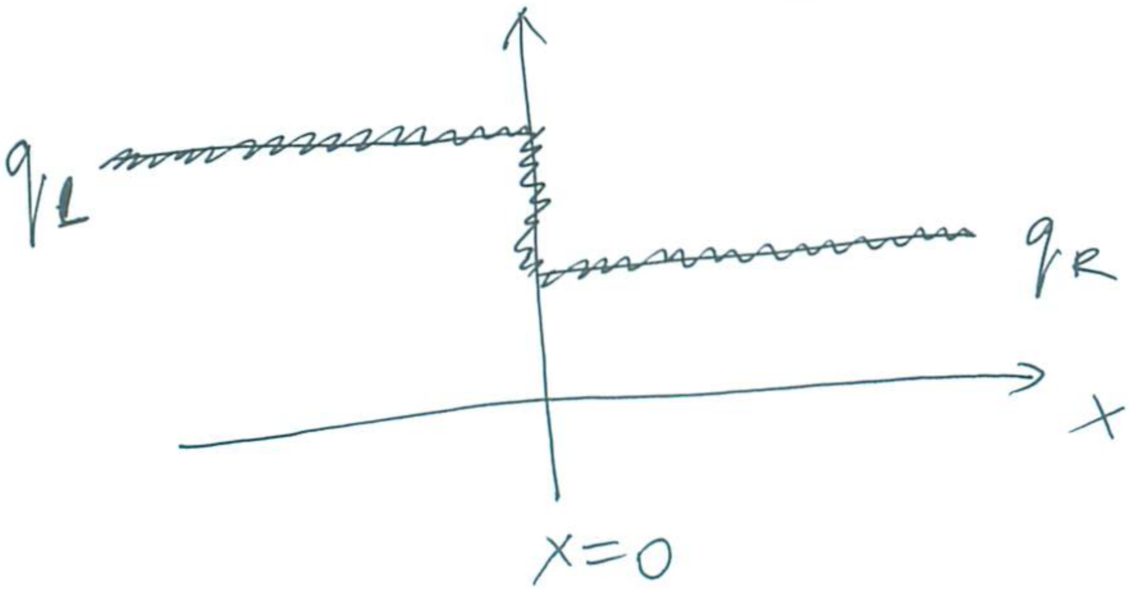
Flux $$\boxed{f_{i+1/2}^n \approx \frac{1}{\Delta t} \int_{t=t^n}^{t=t^n+\Delta t} f\left(q\left(t, x_{i+1/2}\right)\right) dt}$$

How to compute/approximate integral?

Recall GODUNOV'S METHOD based on piecewise-constant reconstruction & solving Riemann problems:



$$\begin{cases} q_L = q_i^n \\ q_R = q_{i+1}^n \end{cases}$$



Note that we only need to solve the Riemann problem for

$$\boxed{q(x=0, t)}$$

Start from scalar one-dimensional hyperbolic equations. (3)

Here the Riemann problem can be solved exactly using the method of characteristics.

$$u_t + \left[f(u) \right]_x = 0$$

$$u_t + a(u)\, u_x = 0 \,, \quad a \equiv \underbrace{\frac{\partial f}{\partial u}}_{\text{Jacobian}}$$

$$\boxed{\begin{array}{l} u(x(t)) = \text{const} = u_0 \\ \text{if } \dfrac{dx}{dt} = a(u_0) \end{array}}$$

characteristic curve

characteristic speed

Characteristics are straight lines even though nonlinear. But, they are not parallel so they can intersect and give rise to contact discontinuities like shocks & rarefaction waves

Conservation (weak form of PDE) dictates that shocks move with speed given by the RANKINE-HYGONIOT condition

$$s = x'(t) = \frac{f(u_R) - f(u_L)}{u_R - u_L}$$

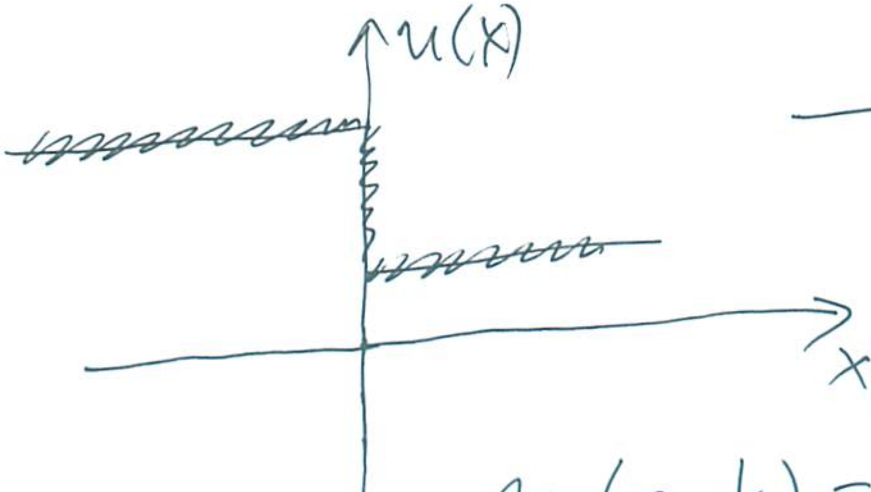
LET'S TAKE Inviscid Burgers

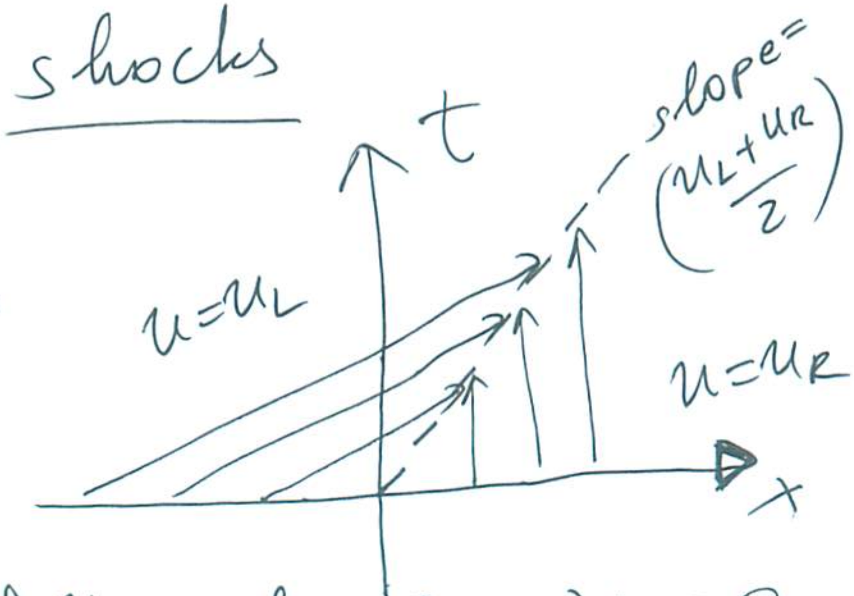
$$u_t + u u_x = u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$f(u) = \frac{u^2}{2}$$

$$s = \frac{u_R^2 - u_L^2}{2(u_R - u_L)} = \frac{1}{2}(u_R + u_L)$$

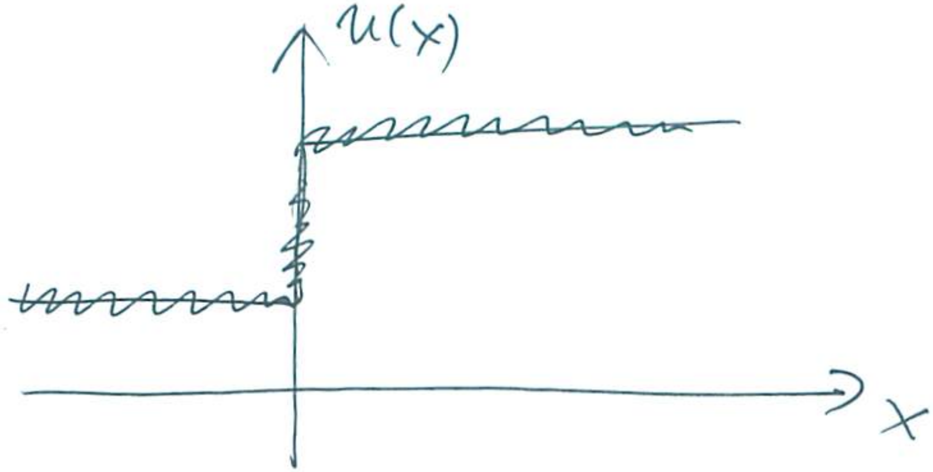
which works for shocks

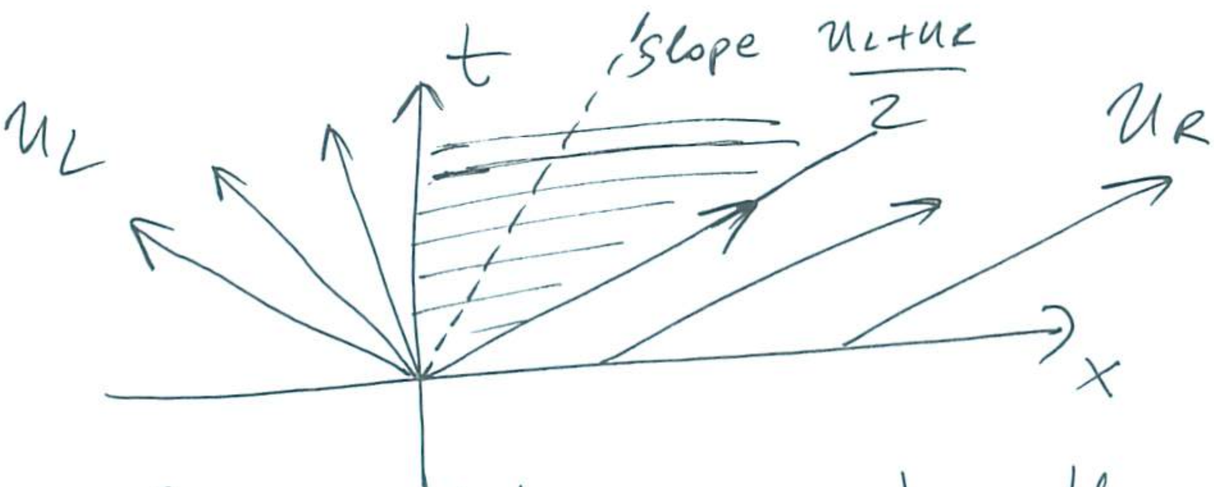




$$u(0,t) = \begin{cases} u_L & \text{if } (u_L + u_R)/2 > 0 \\ u_R & \text{otherwise} \end{cases}$$

There is also <u>rarefaction</u> waves





The solution inside the rarefunction cone can be made unique via an <u>entropy</u> condition and is self-similar

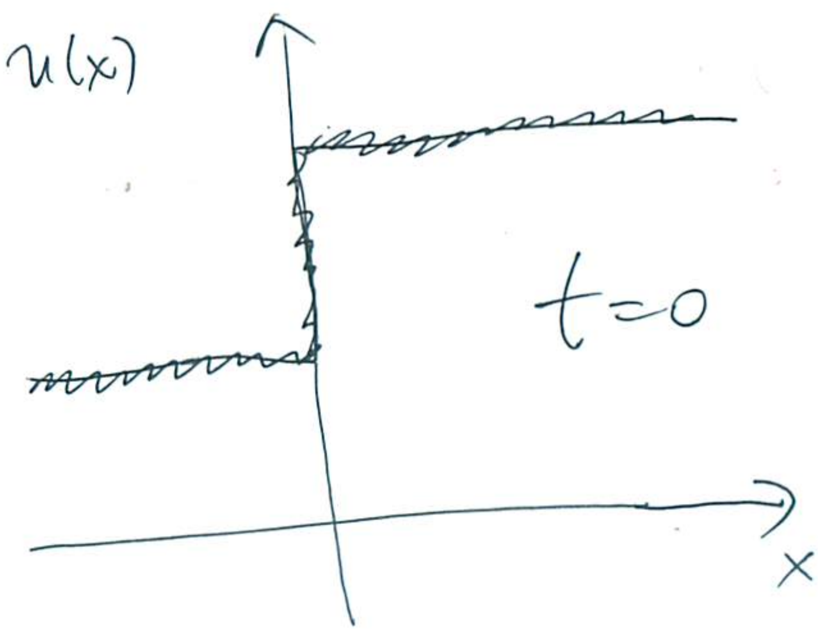
$$\boxed{u(x,t) = g\left(\frac{x}{t}\right)}$$

For Burgers

If $u_L < u_R$

$$u(x,t) = \begin{cases} u_L, & x < u_L t \\ u_R, & x > u_R t \\ x/t & \text{if } u_L t \le x \le u_R t \end{cases}$$

↖ entropy weak solution



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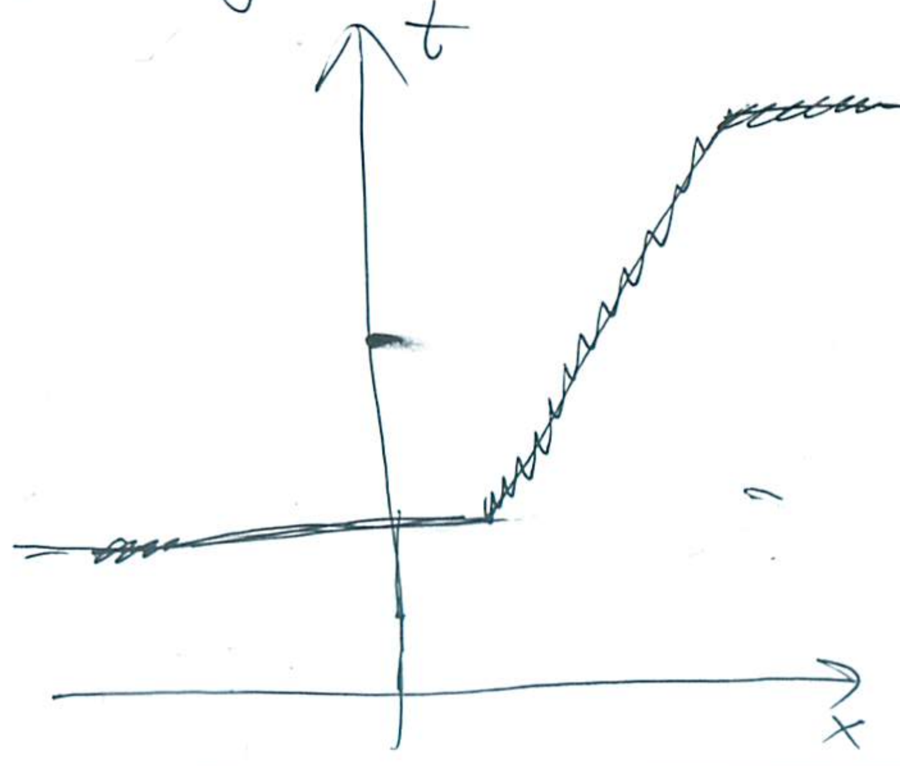

We see here that for $x/t = 0$ (8)
the solution of the Riemann problem is <u>constant</u> in time so the integral of the flux

$$f^{n}_{i+1/2} = \frac{1}{\Delta t} \int \left(\frac{u^{Riemn}(x_{i+1/2}, t)}{2} \right)^2 dt$$

$$f^{n}_{i+1/2} = \left(\frac{u^{Riemam}_{i+1/2}}{2} \right)^2$$

where $u^{Riemam}_{i+1/2} = \begin{cases} u_L \\ u_R \quad \text{or} \\ \emptyset \leftarrow \text{turns out we can ignore this case} \end{cases}$

In the end one can simplify Godunov's scheme to a version of <u>upwinding</u>:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} \begin{cases} (u_{j+1}^n)^2 - (u_j^n)^2 & \text{if } u_j^n < 0 \\ (u_j^n)^2 - (u_{j-1}^n)^2 & \text{if } u_j^n \geq 0 \end{cases}$$

This scheme can be shown to converge to the entropy or viscosity solution but we will not go through that here.

For this to work we must have that characteristics from different cells cannot intersect, i.e., that information does not propagate by more than grid spacing: 10

$$\forall j: \quad \frac{\Delta t \, |a(v_j^n)|}{\Delta x} \leq 1 \qquad \text{CFL condition}$$

One can choose Δt adaptively.

Riemann problems cannot always be solved analytically, though often one can convert them to algebraic equations that can be solved numerically.

An alternative is to use <u>approximate Riemann solvers</u> via <u>linearization</u> (Roe's method and variants).

$$\vec{u}_t + \left[\vec{f}(\vec{u})\right]_x = \vec{0}$$

$$\vec{u}_t + \frac{\partial \vec{f}}{\partial \vec{u}} \cdot \vec{u}_x = \vec{0}$$

Denoting the Jacobian

$$\overleftrightarrow{A} = \partial \vec{f} / \partial \vec{u} \quad \text{at } \underline{\underline{\text{some}}} \text{ state}$$

evaluated

$$\boxed{u_t + A u_x = 0}$$

Linearized equation

Approximate Riemann problem

$$\tilde{u}_t + \tilde{A}^n_{i+1/2} \tilde{u}_x = 0$$

<u>Note</u>: Problem is <u>strictly hyperbolic</u> if A is <u>diagonalizable</u> with all <u>real</u> eigenvalues, distinct if strict.

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E.g.

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$$\tilde{A}^n_{i+1/2} = \frac{\partial \vec{f}}{\partial \vec{u}}\left(\vec{u} = \frac{u_i^n + u_{i+1}^n}{2}\right)$$

ensures that $\tilde{A}$ is diagonalizable with real (distinct) eigenvalues

But another desirable condition

$$\boxed{\tilde{A}_{i+1/2}(u_{i+1} - u_i) = f(u_{i+1}) - f(u_i)}$$

was imposed by Roe.

Note that for Burger's this gives

$$A_{i+1/2} \equiv a_{i+1/2} = \frac{u_i + u_{i+1}}{2} \left(\begin{matrix}\text{shock}\\ \text{speed}\end{matrix}\right)$$

$$u_t + A u_x = 0$$

$$T^{-1} A T = \Lambda = \mathrm{Diag}\{\lambda_i\}$$

real

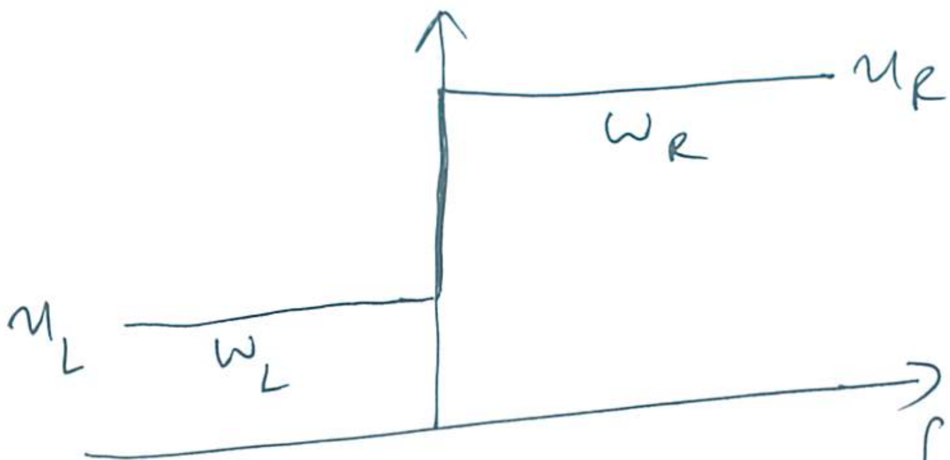
This suggests transforming variables to <u>characteristic variables</u> (called Riemann invariants)

$$w_t + \Lambda w_x = 0$$

so each w_p obeys a <u>simple advection equation</u> with wave speed λ_p

We can use our <u>machinery</u> for advection equations to advect w's!

Approximate Riemann solver



$$w_{L/R} = T^{-1} u_{L/R}$$

$$w_p(0,t) = \begin{cases} w_{L,p} & \text{if } \lambda_p > 0 \\ w_{R,p} & \text{if } \lambda_p \leq 0 \end{cases}$$

so

$$\boxed{u(0,t) = T\, w(0,t)}$$

Read w from left or right cell

$$\boxed{f^n_{i+1/2} = f(u(0,t))}$$

Numerical flux

For Burger's this gives

$$f^n_{i+1/2} = \begin{cases} u_L & \text{if } \frac{u_L+u_R}{2} > 0 \\ u_R & \text{otherwise} \end{cases}$$

which <u>unfortunately fails</u> at "supersonic rarefaction" because it does not converge to the right entropy solution.

Nonlinear hyperbolic PDE's are hard!

One can instead approximate the solution of the exact Riemann problem – we won't discuss this.

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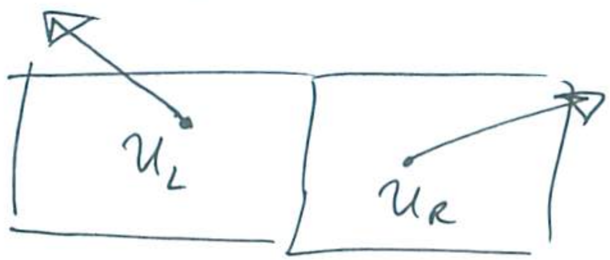
<u>One solution</u>: Compute characteristics from the left <u>and</u> the right.

If they propagate <u>away</u> from the face



(both of them), compute <u>the average of the Riemann invariants</u> traced back to the left & the right.

<u>Note</u>:

One can make this a <u>second-order method</u> by doing <u>linear reconstruction</u> & tracing the characteristics from the linear profile to <u>the midpoint in time</u> (i.e. center in time)

Methods that always work are harder to make higher order, so I just give two simple first-order "upwinding" methods: for scalar case:

(1) Godunov method:

$$f^n_{j+1/2} = \begin{cases} \min\limits_{u_j^n \le \theta \le u_{j+1}^n} f(\theta) & \text{if } u_j^n \le u_{j+1}^n \\ \max\limits_{u_{j+1} \le \theta \le u_j^n} f(\theta) & \text{otherwise} \end{cases}$$

2) Enquist-Osher scheme

$$f^n_{j+1/2} = \frac{f(u_j^n)+f(u_{j+1}^n)}{2} - \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} |f'(\theta)|\, d\theta$$

One can also derive Lax-Wendroff generalizations by second-order expansion replacing space & time derivatives:

$$u_{tt} = -(f(u))_{x,t} = \Big(\underbrace{f'(u)}\,\underbrace{(f(u))_x}_{\text{use centered differences}}\Big)_x$$