Consider numerically solving the advection-diffusion equation

\[ u_t + au_x = (d(x)u_x)_x, \]

on the periodic domain \(0 \leq x < 1\), for \(a = 1\) and initial condition

\[ u(x, 0) = [\sin (\pi x)]^{100}, \]

on a uniform grid, as we did back in Homework 1 for \(d(x) = d = \text{const.}\).

1 **Linear PDE**

[Note: This was an optional part for HW1]

Solve this equation using the Lax-Wendroff or Fromm’s (recommended!) method for the advection, and make the diffusion coefficient non-constant,

\[ d(x) = \epsilon (2 + \cos(2\pi x)). \]

In this case, it is a bit harder to compute an exact solution (please do not try, this is not the point of this homework!). Use a small value for the diffusion coefficient in the range \(\epsilon = 0.01 - 0.05\).

1. Write down a spatio-temporal discretization. Explain how you handled diffusion in your spatio-temporal discretization and what you expect the order of accuracy of the method to be (do your best to make it second-order, of course).

2. Validate your code in some way (e.g., by solving problem with \(d(x) = d = \text{const.}\) and/or using a manufactured analytical solution).

3. Refine the resolution in both space and time (i.e., in space-time, not space or time separately) to empirically estimate the spatio-temporal order of convergence.

4. (Optional) Investigate the stability of your scheme – is it limited in stability by both advection and diffusion or only advection?

   Note: You may find that Fromm’s method behaves differently from Lax-Wendroff.

2 **[Optional] Nonlinear PDE**

Now try your hand at solving the nonlinear equation

\[ u_t + au_x = (d(u)u_x)_x, \]

by repeating the above tasks (as many as you can), using, for example,

\[ d(x) = 0.05 \exp(u). \]