Hydrodynamic fluctuations in quasi-two dimensional diffusion

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1 Diffusion in bulk 2D and 3D
2 Diffusion in Quasi2D
3 Brownian Dynamics in Q2D
4 Numerical Results
Bulk colloidal suspensions in three dimensions (3D) have been studied for a long time.

We consider colloids that are confined by some strong potential to remain on a plane [1]. An example are colloids confined to diffuse on a planar liquid-liquid interface. This has been studied before by Johannes Bleibel, Alvaro Domínguez, and collaborators.

In the limit of strong confining potential, the diffusive dynamics of the colloids is restricted to the plane: quasi two-dimensions (q2D).

Note that the fluid flow around the colloids, mediating hydrodynamic interactions among the particles, is still three dimensional.

If we consider colloids in a very thin film, we have 2D fluid flow: true two-dimensions (t2D).

The goal of this talk will be to study the surprising differences between 3D, t2D and q2D suspensions.
There is a common belief that diffusion in all sorts of materials, including gases, liquids and solids, is described by random walks and **Fick’s law** for the **concentration** of labeled (tracer) particles $c(r, t)$,

$$\partial_t c = \nabla \cdot [\chi(r; c) \nabla c],$$

where $\chi \geq 0$ is a diffusion tensor.

But there is well-known hints that the **microscopic** origin of Fickian diffusion is different in liquids from that in gases or solids, and that **thermal velocity fluctuations** play a key role [2].

The **Stokes-Einstein relation** connects mass diffusion to **momentum diffusion** (viscosity $\eta$) for dilute solutions in 3D,

$$\chi \approx \frac{k_B T}{6\pi\sigma\eta},$$

where $\sigma$ is the tracer (hydrodynamic) diameter.
The thermal velocity fluctuations are described by the (unsteady) fluctuating Stokes equation,
\[ \rho \partial_t v + \nabla \pi = \eta \nabla^2 v + \sqrt{2\eta k_B T} \nabla \cdot W, \quad \text{and} \quad \nabla \cdot v = 0. \quad (1) \]
where the thermal (stochastic) momentum flux is spatio-temporal white noise,
\[ \langle W_{ij}(r, t) W_{kl}^*(r', t') \rangle = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \delta(t - t') \delta(r - r'). \]
The solution of this SPDE is a white-in-space distribution (very far from smooth!).

Define a smooth advection velocity field, \( \nabla \cdot u = 0 \),
\[ u(r, t) = \int \sigma (r - r') v(r', t) \, dr' \equiv \sigma * v, \]
where the smoothing kernel \( \sigma \) filters out features at scales below a molecular cutoff scale \( \sigma \).
Lagrangian description of a passive tracer diffusing in the fluid,
\[ \dot{q} = u(q, t) + \sqrt{2\chi_0} \mathcal{W}_q, \]  
where \( \mathcal{W}_q(t) \) is a collection of white-noise processes (independent among tracers).
In this case \( \sigma \) is the typical size of the tracers.

Eulerian description of the concentration \( c(r, t) \) with an (additive noise) fluctuating advection-diffusion equation,
\[ \partial_t c = -u \cdot \nabla c + \chi_0 \nabla^2 c, \]  
where \( \chi_0 \) is the bare diffusion coefficient.

The two descriptions are equivalent. When \( \chi_0 = 0 \),
\[ c(q(t), t) = c(q(0), 0) \]  
or, due to reversibility,
\[ c(q(0), t) = c(q(t), 0). \]
\[ \rho \partial_t \mathbf{v} + \nabla \pi = \eta \nabla^2 \mathbf{v} + \sqrt{2\eta k_B T} \nabla \cdot \mathbf{W}, \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0. \]

\[ u(r, t) = \int \sigma(r, r') \mathbf{v}(r', t) \, dr' \equiv \sigma \ast \mathbf{v} \]

\[ \partial_t c = -\mathbf{u} \cdot \nabla c + \chi_0 \nabla^2 c \]
Snapshots of concentration in a miscible mixture showing the development of a *rough* diffusive interface due to the effect of *thermal fluctuations*. These *giant fluctuations* have been studied experimentally [3] and with hard-disk molecular dynamics.
In liquids molecules are caged (trapped) for long periods of time as they collide with neighbors: **Momentum and heat diffuse much faster than does mass.**

This means that $\chi \ll \nu$, leading to a **Schmidt number**

$$S_c = \frac{\nu}{\chi} \sim 10^3 - 10^4.$$ 

This **extreme stiffness** solving the concentration/tracer equation numerically challenging.

There exists a **limiting (overdamped) dynamics** for $c$ in the limit $S_c \to \infty$ in the scaling

$$\chi \nu = \text{const.}$$
Adiabatic mode elimination gives the following limiting **stochastic advection-diffusion equation** (reminiscent of the Kraichnan’s model in turbulence),

\[
\frac{\partial}{\partial t} c = -\mathbf{w} \odot \nabla c + \chi_0 \nabla^2 c,
\] (4)

where \( \odot \) denotes a Stratonovich dot product.

The advection velocity \( \mathbf{w}(\mathbf{r}, t) \) is **white in time**, with covariance proportional to a Green-Kubo integral of the velocity auto-correlation function,

\[
\langle \mathbf{w}(\mathbf{r}, t) \otimes \mathbf{w}(\mathbf{r}', t') \rangle = 2 \delta (t - t') \int_0^\infty \langle \mathbf{u}(\mathbf{r}, t) \otimes \mathbf{u}(\mathbf{r}', t + t') \rangle dt',
\]

In the Ito interpretation, there is **enhanced diffusion**,  

\[
\frac{\partial}{\partial t} c = -\mathbf{w} \cdot \nabla c + \chi_0 \nabla^2 c + \nabla \cdot [\chi(\mathbf{r}) \nabla c]
\] (5)

where \( \chi(\mathbf{r}) \) is an **analog of eddy diffusivity** in turbulence.
Stokes-Einstein Relation

- An explicit calculation for Stokes flow gives the explicit result
  \[ \chi(r) = \frac{k_B T}{\eta} \int \sigma(r - r') G(r' - r'') \sigma^T(r - r'') \, dr' \, dr'', \]  
  where \( G \) is the Green’s function for steady Stokes flow.

- For an appropriate filter \( \sigma \), this gives Stokes-Einstein formula for the diffusion coefficient in a finite domain of length \( L \),
  \[ \chi = \frac{k_B T}{\eta} \left\{ \begin{array}{ll}
    (4\pi)^{-1} \ln \frac{L}{\sigma} & \text{if } d = 2 \\
    (6\pi \sigma)^{-1} \left( 1 - \frac{\sqrt{2} \sigma}{2} \frac{\sigma}{L} \right) & \text{if } d = 3.
  \end{array} \right. \]

- The limiting dynamics is a good approximation if the effective Schmidt number \( S_c = \nu/\chi_{\text{eff}} = \nu/(\chi_0 + \chi) \gg 1 \).

- The fact that for many liquids Stokes-Einstein holds as a good approximation implies that \( \chi_0 \ll \chi \): Diffusion in liquids is dominated by advection by thermal velocity fluctuations, and is more similar to eddy diffusion in turbulence than to standard Fickian diffusion.
If we take an **overdamped** limit of the **Lagrangian equation** we get the Ito equations of **Brownian Dynamics** (BD) for the (correlated) positions of the \( N \) particles \( \mathbf{Q}(t) = \{\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t)\} \),

\[
d\mathbf{Q} = \mathbf{M} \cdot \mathbf{F}(\mathbf{Q}) \, dt + (2k_B T \mathbf{M})^{\frac{1}{2}} \, d\mathbf{B} + k_B T (\partial_\mathbf{Q} \cdot \mathbf{M}) \, dt,
\]

where \( \mathbf{B}(t) \) is a vector of Brownian motions, and \( \mathbf{F}(\mathbf{Q}) \) are forces.

Here \( \mathbf{M}(\mathbf{Q}) \succeq 0 \) is a symmetric positive semidefinite (SPD) **mobility matrix**, assumed here to have a far-field **pairwise approximation**

\[
\mathbf{M}_{ij}(\mathbf{Q}) \equiv \mathbf{M}_{ij}(\mathbf{q}_i, \mathbf{q}_j) = \mathcal{R}(\mathbf{q}_i - \mathbf{q}_j),
\]

where \( \mathcal{R} \) is the **hydrodynamic kernel**.

The self-diffusion tensor of a single isolated particle is

\[
\chi = (k_B T) \mathcal{R}(\mathbf{0}).
\]
In our model the hydrodynamic kernel is
\[ \mathcal{R}(r_1 - r_2) = \int \sigma(r_1 - r') \mathcal{G}(r' - r'') \sigma(r_2 - r'') \, dr' \, dr''. \]

Observe that in the far-field, \( r \gg a \), the RPY tensor becomes the long-ranged Oseen tensor
\[ \mathcal{R}(r \gg a) \rightarrow \mathcal{G}(r) = \frac{1}{8\pi r} \left( l + \frac{r \otimes r}{r^2} \right). \] (7)

For 3D bulk suspensions, if \( \sigma(r) = \delta(r - a) \) is a surface delta function, we get the widely-used Rotne-Prager-Yamakawa tensor
\[ \mathcal{R}(r) = \frac{1}{6\pi \eta a} \left( \frac{3a}{4r} + \frac{a^3}{2r^3} \right) l + \left( \frac{3a}{4r} - \frac{3a^3}{2r^3} \right) \frac{r \otimes r}{r^2}, \quad r > 2a. \]
Replace the surface delta function $\delta_a$ by a smooth Gaussian kernel with standard deviation $\sigma = a/\sqrt{\pi}$ to give $\chi = k_B T / (6\pi \eta a)$.

This gives the FCM kernel that is just as good as RPY:

$$\mathcal{R}(r) = f(r) I + g(r) \frac{r \otimes r}{r^2},$$

where

$$\begin{bmatrix} f(r) \\ g(r) \end{bmatrix} = \frac{1}{8\pi \eta r} \left( 1 + \begin{bmatrix} 2 \\ -6 \end{bmatrix} \frac{a^2}{\pi r^2} \right) \text{erf} \left( \frac{r\sqrt{\pi}}{2a} \right).$$

$$- \frac{1}{8\pi \eta r} \begin{bmatrix} 2 \\ -6 \end{bmatrix} \frac{a}{\pi r} \exp \left( -\frac{\pi r^2}{4a^2} \right).$$

The use of FHD (fluctuating hydrodynamics) with Gaussian kernels allows for very efficient (linear time!) BD, even for the RPY kernel [4].
\[ dQ = \mathbf{M} \cdot \mathbf{F}(Q) \, dt + (2k_B T \mathbf{M})^{\frac{1}{2}} \, d\mathcal{B} + k_B T (\partial_Q \cdot \mathbf{M}) \, dt. \]

- An important property of the 3D RPY and FCM kernel is that they are divergence free,
  \[ \nabla \cdot \mathcal{R}_{3D}(\mathbf{r}) = 0, \]
  which follows from the fact the 3D flow is incompressible,
  \[ \nabla \cdot \mathcal{G}(\mathbf{r}) = 0, \]
  and implies that
  \[ \partial_Q \cdot \mathbf{M} = 0. \]
  This has important consequences on collective diffusion.

- The same applies for t2D systems as well,
  \[ \nabla \cdot \mathcal{R}_{t2D}(\mathbf{r}) = 0, \]
  but there are still some important differences between t2D and 3D diffusion related to giant fluctuations.
For q2D, dynamics can be described by BD-HI with \( \mathbf{q} = (x, y) \) being position in the plane.

Now the hydrodynamic kernel is still the same RPY or FCM kernel, but now the flow is not incompressible in the plane,
\[
\nabla_{(x,y)} \cdot \mathcal{R}_{q2D}(r) \neq 0,
\]
which means that there will be a nonzero \( \partial \mathbf{Q} \cdot \mathbf{M} \), and the diffusive dynamics will be very different from either 3D or t2D.

To start take the Oseen tensor as the hydrodynamic kernel,
\[
f(r \gg a) \approx g(r \gg a) \approx \frac{1}{8\pi \eta r},
\]
which gives something that in the far field looks like a repulsive Coulomb force,
\[
\frac{dq_i}{dt} = \cdots + k_B T (\partial \mathbf{Q} \cdot \mathbf{M})_i = \cdots + \sum_{j \neq i} \frac{k_B T}{8\pi \eta r} \cdot \frac{\mathbf{q}_i - \mathbf{q}_j}{\| \mathbf{q}_i - \mathbf{q}_j \|^2} + \cdots
\]
For the majority of the rest of this talk we assume particles do not interact with a direct potential (ideal gas). Unphysical but steric repulsion does not change (short-time) collective diffusion that much.

Define a concentration from the positions of the particles \( q_i(t) \),

\[
c(r, t) = \sum_{i=1}^{N} \delta(q_i(t) - r),
\]

(8)

Ito’s rule gives the following (formal) closed but nonlinear stochastic advection-diffusion equation for the concentration [5],

\[
\partial_t c(r, t) = \nabla \cdot (\chi(r) \nabla c(r, t)) - \nabla \cdot (w(r, t) c(r, t)) \\
+ (k_B T) \nabla \cdot \left( c(r, t) \int \mathcal{R}(r, r') \nabla' c(r', t) \, dr' \right).
\]

(9)

Fluctuations come via the random velocity field \( w \) that comes from the fluctuating fluid velocity in FHD.
Nonlocal (Far-Field) HIs in 3D/t2D

- The **nonlinear nonlocal hydrodynamic** term can be rewritten as
  \[
  \nabla \cdot \left( c(r, t) \int \mathcal{R}(r, r') \nabla' c(r', t) \, dr' \right) = \\
  - \nabla \cdot \left( c(r, t) \int (\nabla' \cdot \mathcal{R}(r, r')) c(r', t) \, dr' \right).
  \]

- For 3D and t2D, \( \nabla \cdot \mathcal{R}(r, r') = \nabla' \cdot \mathcal{R}(r, r') = 0 \), and (9) becomes a **linear** stochastic equation that can easily be solved numerically.

- Importantly, in 3D/t2D, we get Fick’s law even with HIs [2]:
  \[
  \partial_t c^{(1)}(r, t) = \nabla \cdot \left( \chi(r) \nabla c^{(1)}(r, t) \right),
  \]
  for the single-particle distribution function \( c^{(1)}(r, t) = \langle c(r, t) \rangle \).

- But the story is not so simple if one looks at **giant fluctuations**, as I will show later and has been measured in 3D experiments.
The story is very different in q2D because now $\nabla \cdot \mathcal{R}(r) \neq 0$ and it is long-ranged, giving

$$
\partial_t c^{(1)}(r, t) = \nabla \cdot \left( \chi(r) \nabla c^{(1)}(r, t) \right) + (k_B T) \nabla \cdot \left( \int \mathcal{R}(r, r') \nabla' c^{(2)}(r, r', t) \, dr' \right),
$$

which is not closed, is nonlocal, and nonlinear.

For an ideal gas, the standard closure for the two-particle correlation function is

$$
c^{(2)}(r, r', t) \approx c^{(1)}(r, t) c^{(1)}(r', t),
$$
giving the approximation

$$
\partial_t c^{(1)}(r, t) = \nabla \cdot \left( \chi(r) \nabla c^{(1)}(r, t) \right) .
$$

$$
+(k_B T) \nabla \cdot \left( c^{(1)}(r, t) \int \mathcal{R}(r, r') \nabla' c^{(1)}(r', t) \, dr' \right)
$$
Dynamics of Density Fluctuations in q2D

- Consider the case of a spatially uniform system with concentration $c(r, t) = c_0 + \delta c(r, t)$, where $\delta c \ll c_0$.

- If we linearize (9) around the uniform state and ignore fluctuations:
  \[
  \partial_t \delta c(r, t) = \chi \nabla^2 \delta c(r, t) + (k_B T) \nabla \cdot \left( c_0 \int \mathcal{R}(r-r') \nabla' \delta c(r', t) \, dr' \right).
  \]

- This equation can trivially be solved in Fourier space,
  \[
  \frac{d}{dt} (\hat{\delta c}_k) = - \left( \chi k^2 + (k_B T) c_0 k \cdot \hat{\mathcal{R}}_k \cdot k \right) \hat{\delta c}_k = -\chi k^2 D_c(k) \hat{\delta c}_k,
  \]
  where $D_c(k)$ is the short-time collective diffusion coefficient,
  \[
  D_c(k) = \chi \left( 1 + \frac{1}{kL_h} \right) = \chi + (k_B T) \frac{c_0}{4\eta k}.
  \]

- For high packing densities $\phi = \pi c_0 a^2 \sim 1$, we have $L_h \sim a$: strong collective diffusion effects at all length scales.
By combining the Fluctuating Immersed Boundary (FIB) method with the Fluctuating Force Coupling Method (FCM) we obtain an efficient $O(N)$ algorithm for q2D-BD.

The key idea behind both of these is to use fluctuating hydrodynamics to obtain the random displacements but I will present it here from a more algebraic perspective [4].

The key is to go Fourier space, with $\kappa = (k, k_z)$,

$$\hat{R}_k = \frac{1}{2\pi} \int_{k_z} dk_z \frac{dk_z}{\eta k_z^2} \left( I - \frac{\kappa \otimes \kappa}{k_z^2} \right) \exp \left( -\frac{a^2 k_z^2}{\pi} \right).$$

$$= \frac{1}{\eta k^3} \left( c_2 (ka) \ k_\perp \otimes k_\perp^T + c_1 (ka) \ k \otimes k^T \right). \quad (13)$$

where both $c_1$ and $c_2$ decay exponentially $\sim \exp (-a^2 k^2)$ in Fourier space (pseudospectral methods).
For small $k$ we have the 2D projection of the t2D or q2D Oseen tensor,

$$c_1 (K = ka \ll 1) \approx \frac{1}{4} \text{ for q2D, and } 0 \text{ for t2D, and}$$

$$c_2 (K = ka \ll 1) \approx \frac{1}{2} \text{ for q2D, and } \frac{1}{k} \text{ for t2D.}$$

The short-time self diffusion coefficient $\chi_0 = f \left( k_B T / \eta \right)$,

$$f = \frac{1}{6\pi a} \cdot \frac{1}{1 + 4.41a/L} \approx \frac{1}{6\pi a} \text{ for q2D, and}$$

$$f = \frac{1}{4\pi} \ln \left( \frac{L}{3.71a} \right) \text{ for t2D,}$$

and $L$ is the system size.
Brownian Dynamics in Q2D

Diffusion as random advection

- For an **ideal gas** we have the Ito BD equation:
  \[ dQ = (2k_B T M)^{\frac{1}{2}} dB + k_B T (\partial Q \cdot M) \, dt, \]  
  \[ (15) \]

- Brownian motion of a particle in an ideal gas in q2D [5]:
  \[ \frac{dq_i}{dt} = w(q_i, t) + k_B T \left( a(q_i) + \sum_{j \neq i} b(q_i, q_j) \right), \]  
  \[ (16) \]

  where \( a(r) = \nabla \cdot R(r, r) = \nabla \cdot \chi(r) \) and \( b(r, r') = \nabla' \cdot R(r, r') \).

- For a translationally-invariant system \( a = 0 \), and for t2D \( b = 0 \).

- Here \( w(r, t) \) is a **random fluid velocity** that advects the particles,
  \[ \langle w(r, t) \otimes w(r', t') \rangle = 2 (k_B T) R(r, r') \delta(t - t') . \]  
  \[ (17) \]
The final BD equation is, with \( \partial_i \delta_a (r) = \partial \delta_a (r) / \partial r_i \) [5],

\[
\frac{d q_i}{dt} = w(q_i, t) + \int \delta_a (q_i - r') \sum_j G(r', r'') \, dr' \, dr'' .
\]  

(18)

\[
[F_j \delta_a (q_j - r'') + (k_B T) (\partial \delta_a) (q_j - r'')] .
\]

From (13) we get

\[
\hat{w}_k = \sqrt{\frac{2k_B T}{\eta k^3}} \left( \sqrt{c_2 (ka)} \, k_\perp \mathcal{Z}_k^{(2)} + \sqrt{c_1 (ka)} \, k \mathcal{Z}_k^{(1)} \right),
\]  

(19)

where \( \mathcal{Z}_k^{(1/2)} (t) \) are independent white noise processes – stochastic momentum flux in fluctuating Stokes equation.

For FCM the kernel \( \delta_a \) is a Gaussian with \( \sigma = a / \sqrt{\pi} \),

\[
\hat{G}_k = \hat{R}_k \exp \left( \frac{a^2 k^2}{\pi} \right) = \frac{1}{\eta} \left[ g_k (k) \, k_\perp \otimes k_\perp^T + f_k (k) \, k \otimes k^T \right].
\]
1. Evaluate particle forces $F^n = F(Q^n)$.

2. Compute in real space on a grid the fluid forcing

$$f(r) = \sum_i F_i \delta_a (q_i - r) + (k_B T) \sum_i (\partial \delta_a)(q_i - r).$$

and use the FFT to convert $f$ to Fourier space, $\hat{f}_k$.

3. Compute the fluid velocity resulting from fluid forcing $f$ in Fourier space as a convolution with the Green’s function,

$$\hat{v}_{det}^k = \hat{G}_k \hat{f}_k.$$
1. Generate a random fluid velocity with covariance \((2k_B T) \hat{G}_k\) in Fourier space,

\[
\hat{v}_{k}^{\text{stoch}} = \sqrt{\frac{2k_B T}{\eta \Delta t}} \left( \sqrt{g_k(k)} k_{\perp} Z_k^{(2)} + \sqrt{f_k(k)} k Z_k^{(1)} \right).
\]

2. Use the FFT to compute \(v(r)\) from

\[
\hat{v}_k = \hat{v}_k^{\text{det}} + \hat{v}_k^{\text{stoch}}.
\]

3. Convolve \(v(r)\) with a Gaussian in real space to compute particle velocities,

\[
u_i = \int \delta_a(q_i - r) v(r) \, dr.
\]

4. Advance the particles,

\[
q_i^{n+1} = q_i^n + u_i \Delta t.
\]
Collective diffusion coefficient

Figure: Short time collective diffusion coefficient in q2D obtained from the dynamic structure factor (autocorrelation function of the spatial FFT).
Numerical Results

Relaxation of density bump (instance)

**Figure:** Expansion of clump in Quasi2D (top) and True2D (bottom). Compare fluctuations for classical diffusion BD-noHI to True2D.
Figure: Comparison of ensemble average to (numerical) DDFT-HI.
If we **color the particles** red and green, \( c^{(1)} = c^{(1)}_R + c^{(1)}_G \), we expect:

\[
\frac{\partial}{\partial t} c^{(1)}_{R/G}(\mathbf{r}, t) = \nabla \cdot \left( \chi \nabla c^{(1)}_{R/G}(\mathbf{r}, t) \right) + (k_B T) \nabla \cdot \left( c^{(1)}_{R/G}(\mathbf{r}, t) \int \mathcal{R}(\mathbf{r}, \mathbf{r}') \nabla' \left( c^{(1)}_R(\mathbf{r}', t) + c^{(1)}_G(\mathbf{r}', t) \right) \, d\mathbf{r}' \right)
\]

If we start the system with a uniform density, \( c^{(1)} = c^{(1)}_R + c^{(1)}_G = c_0 \), this will remain the case forever and we just get two uncoupled diffusion equations

\[
\frac{\partial}{\partial t} c^{(1)}_{R/G}(\mathbf{r}, t) = \nabla \cdot \left( \chi \nabla c^{(1)}_{R/G}(\mathbf{r}, t) \right).
\]

This means that **diffusive mixing** in q2D, is the same on average as for simple BD-noH1 (uncorrelated Brownian walkers) and t2D. But the **fluctuations are different**.
Figure: Color diffusion in q2D (left) versus t2D (right) (100K particles, $\phi \approx 1$).
Figure: Diffusion of a perturbation of color (no-HI, q2D, and t2D)
Numerical Results

Giant Color Fluctuations in t2D

Figure: Giant fluctuations in q2D compared to linearized FHD theory.

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Numerical Results

Giant Color Fluctuations in q2D

Figure: Giant fluctuations in q2D compared to linearized FHD theory.

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Conclusions/questions

1. Diffusion is very strongly affected by hydrodynamic correlations and its nature depends heavily on the geometry of the fluid and the diffusion manifold.

2. In true-2D (diffusion in thin films) the mean obeys simple Fick’s law at all scales but the fluctuations are giant.

3. In quasi-2D (diffusion on flat interfaces) the fluctuations are not giant but the mean does not obey Fick’s law (at any scale?).

4. How are lipid membranes different: At what scales does the Saffman kernel work?

5. What is the long-time collective diffusion coefficient in q2D? Does a generalized Einstein-relation relating a “Fick” coefficient to collective mobility and isothermal compressibility hold?

6. How about diffusion of colloids on a sphere?
Hydrodynamic fluctuations in quasi-two dimensional diffusion.
Software available at https://github.com/fbusabiaga/fluam and
https://github.com/stochasticHydroTools/FHDq2D.

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