Modeling Systemic Risk in The Options Market

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Problem: Dimension Reduction

- How do we model the options market as a whole?

- Options data is 130 times bigger than equity data: we have about 7000 optionable stocks, with 130 options on each. The correlation matrix alone is $910,000 \times 910,000$.

- Computationally costly to perform standard analysis such as PCA on portfolios of options, especially on a daily or on a regular basis.

- Find a way to model the IVS while reducing dimension and preserving information.
Problem: Significant Components in the Options Market as a Whole

- Use developed model to perform PCA analysis on the entire equities and options market\(^1\).

- Examine suitability of RMT techniques in separating signal from noise in the resulting eigenvalues.

- Classify significant factors in the equities and options market as a whole.

\(^1\)As made available by WRDS’s OptionMetrics Database.
Initial Steps

- The underlying asset is a critical component of the option itself. So we couple the implied volatility returns across different deltas and expirations with those of the underlying.

- This results in a $T$ by $N$ data matrix $X$ of call-options.

- $T$ goes back to asset creation date or August 31, 2004, until August 31, 2013.

- We fix $N$ at 53: the asset returns and 52 volatility returns determined by $\delta \in \{20, 25, \ldots, 75, 80\}$, and expiration $\in \{30, 91, 182, 365\}$. 
Introduction

- It is known that the IVS moves in time and exhibits both time and delta skew. We wish to gain a better understanding of these movements via historical analysis of IVSs.

- We normalize all returns since we wish to consider all information as co-movements of variables, regardless of their variance. We begin by performing a PCA on the correlation matrix C of X.

- We focus on the statistical features our model must account for by identifying significant eigenvalues within the framework of Marchenko-Pastur.

- We perform this study for each asset in the WRDS OptionMetrics database. This results in about 3800 assets in total.
Principal Component Surfaces for the S&P500

First four principal components of the implied volatility surface for SPX.
First four PCs of the IVS can be interpreted as being respectively:

1. A parallel shift in the entire surface
2. Delta-slope and time-to-maturity slope
3. Time-to-maturity convexity

The first PC explains the vast majority of the movement of the IVS with a corresponding eigenvalue of about 92 percent. It allocates roughly equal weight to all volatilities, and the corresponding surface is flat.

The first three PCs alone account for roughly 97 percent of the variation in the surface.

The second and third PCs exhibit both time and delta skew. Combined they account for 5 percent of the variation.

The fourth component is flat and is largely insignificant.
Results for the top 20 most liquid ETFs

- We perform the same analysis for the top 20 most liquid ETFs by market volume.

- The results are very comparable to what we observed with SPX: the top three eigenvalues suffice to capture the majority of variation in the IVS.

- The average values for the top 20 most liquid ETFs for the first, second, and third eigenvalues are 76, 6.6, and 4.4 percent respectively.

- With an average of 87.2 percent of variation explained by the top three components.
Variation explained by the first component: top 20 most liquid ETFs

Percent of variation explained by the first component for the 20 most liquid ETFs listed.
Some ETFs such as EWJ, TLT, XLP, XLU, and VXX have more than 1 standard deviation less explanatory power in their first PC.

Furthermore, the second, third and fourth eigenvalues for these outliers are almost two standard deviations above the mean.

- A weaker signal from the first PC, and stronger signals from the second and third components indicate stronger idiosyncratic risk.
- EWJ carries a lot of uncertainty and speculation as it is tied to the Japanese economy and monetary policy. Who knows what they’re doing?
- Compare to the very large explanatory power of the first PC for SPX, which is characterized only by systemic risk.
Quick Overview of Random Matrix Theory: Marchenko-Pastur Distribution

- Let $X$ denote an $M \times N$ random matrix whose entries are i.i.d. with mean 0 and variance $\sigma^2 < \infty$. Denote the correlation matrix and the spectrum (viewed as random variables) by:

  $$Y_N = \frac{1}{N}XX' \text{ and } \{\lambda_1, \lambda_2, \cdots, \lambda_n\}.$$

- Consider the density of states (dos):

  $$\mu_M(A) := \frac{1}{M} \#\{\lambda_j \in A\}, A \subset \mathbb{R}.$$ 

**Theorem**

**MP-distribution:** Assume $M, N \to \infty$ s.t. $\frac{M}{N} \to \Lambda \in (0, \infty)$. Then $\mu_N \to \mu$ in distribution, where

$$\mu(A) = \begin{cases} 
(1 - \frac{1}{\Lambda})1_{0 \in A} + \nu(A), & \text{if } \Lambda > 1 \\
\nu(A), & \text{if } 0 \leq \Lambda \leq 1
\end{cases}$$
Quick Overview of Random Matrix Theory: Marchenko-Pastur Distribution

- and

\[ d\nu(x) := \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x\Lambda} 1_{[\lambda_-,\lambda_+]} dx. \]

- Where \( \lambda_\pm = \sigma^2(1 \pm \sqrt{\bar{\Lambda}})^2. \)

- Define the \textit{MP-threshold} to be the upper bound \( \lambda_+ := (1 + \sqrt{\frac{N}{T}})^2. \) Where \( N = 53 \) corresponds to the number of variables we use, and \( T = \max(\text{asset creation date, August 2004}). \)
The top three eigenvalues are already so distinguished; regardless, wish to see what RMT has to say w.r.t. significance.

For each option in OptionMetrics we generate a random matrix $R$ with the same underlying distribution as the empirical data by permuting the time-series for each of the 53 variables (independently of each other).

We compute the correlation matrix $C$ of $R$, and compute the spectrum of $C$.

We compute the empirical CDF $F_n(x) := \frac{1}{n} \sum_{i=1}^{n} (1_{X_i \leq x})$ where $X_i$ represents the $i$th eigenvalue.
Does Marchenko-Pastur Apply to our Data?

- Using Kolmogorov-Smirnov, we calculate the test statistic
  
  \[ D_n = \sqrt{n} \sup_x |F_n(x) - CMP(x)| \]

  where \( CMP(x) \) denotes the cumulative MP distribution.

- We test the null hypothesis that the empirical data is generated from its corresponding MP-distribution.

- We find that for 98 percent of all option contracts (3800) we cannot reject the null hypothesis at the 1 percent significance level.

- For those options with underlying in S&P500 (440), we find that for 100 percent of the option contracts we cannot reject the null hypothesis at the 1 percent significance level.

- I.e., the distribution of the spectrum of a random correlation matrix whose underlying data has the same distribution as our original data agrees with the MP-distribution.
We compute $C$ the Pearson estimator of the correlation matrix:

$$ C = \frac{1}{T} X' X. $$

We denote by “significant” all eigenvalues greater than $\lambda_+$. These eigenvalues are useful in separating signal from noise.

On average three or four of the eigenvalues and their corresponding eigenvectors lie outside of the Marchenko-Pastur upper bound.

This gives further support that the top three or four eigenvalues suffice in describing change in the IVS.
Marchenko-Pastur threshold $\lambda_+$ as signal indicator

Number of eigenvalues exceeding the Marcenko-Pastur upper bound

Number of eigenvalues exceeding the MP upper limit of the spectrum for the 20 most liquid ETFs.
The Marchenko-Pastur threshold gives us another way of gauging idiosyncratic and systemic risk.

Notice the much higher number of significant eigenvalues corresponding to ETFs like EWJ.

Likewise note the very small number of eigenvalues exceeding $\lambda_+$ for SPX which is characterized by systemic risk.

There is strong negative correlation between the magnitude of the leading eigenvalue and the number of eigenvalues greater than $\lambda_+$. 
The first eigenvalue can be used to measure systemic risk.

It is strongly negatively correlated to the second, third, and fourth eigenvalues, as well as to the number of eigenvalues exceeding the Marchenko-Pastur bound.

The second, third, and fourth eigenvalues are positively correlated to each other, and can be used to measure idiosyncratic risk.
Percent of variation explained by the leading eigenvalue for each constituent of the S&P500. Average value is 68 percent. Average value for the second eigenvalue is 7.6 percent.
Percent of variation explained by the first three principal components for each constituent of the S&P500. Average value is 82 percent. Average number of eigenvalues exceeding $\lambda_+$ is 4.3.
Another look at the constituents of the S&P500

Table: Top 7 constituents and bottom 7 constituents by first eigenvalue.

<table>
<thead>
<tr>
<th>Bottom 7</th>
<th>EV1</th>
<th>Top 7</th>
<th>EV1</th>
</tr>
</thead>
<tbody>
<tr>
<td>KMI</td>
<td>0.27272</td>
<td>GS</td>
<td>0.87176</td>
</tr>
<tr>
<td>POM</td>
<td>0.31406</td>
<td>JPM</td>
<td>0.87092</td>
</tr>
<tr>
<td>WEC</td>
<td>0.34958</td>
<td>BAC</td>
<td>0.85115</td>
</tr>
<tr>
<td>PNW</td>
<td>0.35611</td>
<td>SLB</td>
<td>0.84669</td>
</tr>
<tr>
<td>HCBK</td>
<td>0.35995</td>
<td>CAT</td>
<td>0.84219</td>
</tr>
<tr>
<td>NLSN</td>
<td>0.36338</td>
<td>AAPL</td>
<td>0.84126</td>
</tr>
<tr>
<td>TE</td>
<td>0.36342</td>
<td>XOM</td>
<td>0.84109</td>
</tr>
</tbody>
</table>

- Options with underlying assets in the top 7 face strong systemic risk. Those with underlying assets in the bottom 7 face strong idiosyncratic risk.
Overview of Results so Far

- We can model the change in the IVS for any option by using three main effects due to level, skew, and slope. This is all we need to account for the majority of variation in the surface.

- The leading eigenvalue plays a critical role in classifying option risk into two main categories: systemic risk and idiosyncratic risk.

- Options on large and popular stocks, indexes and certain ETFs tied to the economy and heavily correlated to S&P500 carry mostly systemic risk.

- Options on smaller, more obscure, and more specific names face mostly idiosyncratic risk.

- The leading eigenvalue is usually very high, over 70 percent in most cases. This high degree of correlation among the variables give us hope that we may be able to model the IVS using only a few IVOL returns.
What does the equities market for all constituents of S&P500 look like? What about its options market?

Better yet, what does the entire equities market look like? What about the entire options market? How many significant components are there? Which tools can we use to determine significance?

Due to limitations in computing we cannot use all implied volatility points to model the IVS of such large markets.

Consider modeling the IVS with less pivots, thus providing useful dimensionality reduction, and a possible solution to analysing big data.
Seven Pivot Models for Simulating the IVS

- Use specific implied volatility returns as pivots to get all other implied volatility returns via interpolation.

<table>
<thead>
<tr>
<th></th>
<th>2-Pivots</th>
<th>4-Pivots</th>
<th>5-Pivots</th>
<th>6-Pivots</th>
<th>7-Pivot</th>
<th>9-Pivots</th>
<th>12-Pivots</th>
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</thead>
<tbody>
<tr>
<td>25δ 30</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50δ 30</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td></td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>75δ 30</td>
<td></td>
<td>YES</td>
<td>YES</td>
<td></td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>25δ 91</td>
<td>YES</td>
<td></td>
<td>YES</td>
<td></td>
<td>YES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50δ 91</td>
<td></td>
<td>YES</td>
<td></td>
<td></td>
<td>YES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>75δ 91</td>
<td>YES</td>
<td></td>
<td></td>
<td></td>
<td>YES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25δ 182</td>
<td>YES</td>
<td>YES</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>50δ 182</td>
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<td>YES</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>75δ 182</td>
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<tr>
<td>25δ 365</td>
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<td>YES</td>
<td></td>
<td></td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>50δ 365</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td></td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>75δ 365</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
</tbody>
</table>
Schema of linear interpolation amongst the pivots via projection onto $\delta$-space and expiration-space.
Performance of the 5-Pivot Model: Preserving Top Eigenvalues

Difference between the top two eigenvalues of the 5-pivot model and those of the original spectra. Note that the x-axis is the list of all constituents of SPX in increasing order of their first eigenvalue as computed from the original data using all 52 implied volatility returns.
Difference between the top two eigenvalues of the 6-pivot model and those of the original spectra. Note that the x-axis is the list of all constituents of SPX in increasing order of their first eigenvalue as computed from the original data using all 52 implied volatility returns.
Performance Across of the 9-Pivot Model: Preserving Top Eigenvalues

Difference between the top two eigenvalues of the 9-pivot model and those of the original spectra. Note that the x-axis is the list of all constituents of SPX in increasing order of their first eigenvalue as computed from the original data using all 52 implied volatility returns.
Performance Across the Various Pivot-Models

- The constituent stocks are ordered by increasing original leading eigenvalue, and all differences tend to zero $\Rightarrow$ the performance of any model is greatly improved for options characterized by systemic risk.

- Otherwise said, a model with many pivots becomes more important for modeling the IVS of those options characterized mainly by idiosyncratic risk.

- We discover that during very volatile periods, the leading eigenvalue for any asset tends to increase, indicating overall higher systemic risk; hence the performance of any model improves during such periods.
Original distribution across the various risk-classes using all 52-pivots for options on the constituents of S&P500 vs. the distribution produced by each model indicated. Overall we use the 9-pivot model.
We wish to distinguish signal from noise as determined by the spectrum. Where does the random bulk of eigenvalues end, and where does the signal-carrying portion begin?

Tracy-Widom: The distribution of the largest eigenvalue, $\lambda_{\text{max}}$, of a random correlation matrix is given by:

$$Pr( T\lambda_{\text{max}} < \mu_{TN} + s\sigma_{TN} ) = F_1(s)$$

with

$$\mu_{TN} = (\sqrt{T-.5} + \sqrt{N-.5})^2$$

and

$$\sigma_{TN} = (\sqrt{T-.5} + \sqrt{N-.5})\left(\frac{1}{\sqrt{T-.5}} + \frac{1}{\sqrt{N-.5}}\right)^{\frac{1}{3}}$$
Tracy-Widom CDF for the $\beta = 1$ gaussian orthogonal ensemble case. 100 percentile $\rightarrow s=13.63$, 99 percentile $\rightarrow s=2.06$, and 95 percentile $\rightarrow s=1$. 
Impact of Fat Tails

- Tracy-Widom holds for specific $\beta$-ensembles: Gaussian Orthogonal Ensemble: $\beta = 1$, Gaussian Unitary Ensemble: $\beta = 2$, and Gaussian Symplectic Ensemble: $\beta = 4$. We use $\beta = 1$.

- How do things change for fat-tailed distributions? Can we still use Tracy-Widom?

- It has been shown [Bouchaud et. al.] that fat-tails can massively increase the maximum eigenvalue in the theoretical limiting spectrum of the random matrix.

- What happens when we apply Tracy-Widom to a random matrix with the same distribution as our data?
Generate time-series permutations of our original data and calculate the largest eigenvalue corresponding to its correlation matrix.

Repeat 10,000 times, and calculate the maximum eigenvalue each time.

It turns out there is excellent agreement between Marchenko-Pastur’s $\lambda_+$ and our $\tilde{\lambda}_+$. In addition, the corresponding $s$-value is in very good agreement with Tracy-Widom.

Based on this experiment, let’s see where Tracy-Widom says the significant bulk of eigenvalues begins for our original data.
91 percent of generated maximum eigenvalues lie below the theoretical value of $\lambda_+$, and all simulated maximum eigenvalues lie in [2.08, 2.22].
Empirical density of spectrum of random correlation matrix of options on S&P500 vs. MP. For both distributions there is a point mass of weight 54 percent at zero. We cannot reject the null hypothesis that the two distributions are the same based on Kolmogorov-Smirnov 2-series test at the 1 percent significance level.
CDF of largest eigenvalue of random correlation matrix on all equities vs. Tracy-Widom. We cannot reject the null hypothesis that the two distributions are the same based on Kolmogorov-Smirnov one-series test at the 1 percent significance level. Overall 10,000 simulations were performed and the corresponding eigenvalue in each case was computed.
Table: *Significant Eigenvalues in the Entire Options Market a la Tracy-Widom*

<table>
<thead>
<tr>
<th>Top 110 Eigenvalues</th>
<th>s-value</th>
<th>$F_1(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 = 3742$</td>
<td>24843</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_5 = 209.27$</td>
<td>879.14</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{10} = 143.5$</td>
<td>433.04</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{20} = 118.19$</td>
<td>261.32</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{40} = 102.62$</td>
<td>155.74</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{50} = 97.40$</td>
<td>120.35</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{70} = 90.48$</td>
<td>73.35</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{90} = 84.56$</td>
<td>33.21</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_{107} = 80.21$</td>
<td>3.70</td>
<td>.9996</td>
</tr>
<tr>
<td>$\lambda_{108} = 80.04$</td>
<td>2.60</td>
<td>.996</td>
</tr>
<tr>
<td>$\lambda_{109} = 79.65$</td>
<td>-.10</td>
<td>.80</td>
</tr>
<tr>
<td>$\lambda_{110} = 79.41$</td>
<td>-1.71</td>
<td>.35</td>
</tr>
</tbody>
</table>

- 31410 assets and 500 days used: corresponding $\lambda_+ = 79.672$.
- 108 eigenvalues exceed it and account for 50 percent of variation.
- All 108 are deemed significant by Tracy-Widom.
Conclusion and Takeaways: Pivot model and applicability of RMT-techniques

▶ We can model the IVS using 9-pivots; this model provides a reduction of over 14 times in the number of variables needed.

▶ We test the applicability of Marchenko-Pastur and Tracy-Widom on random matrixes with the same underlying distribution as our empirical data.

▶ We find that the results produced from using such random matrixes are in excellent agreement with those predicted by Marchenko-Pastur.

▶ The results are in very good agreement with those predicted by Tracy-Widom: the underlying distribution of our empirical data\(^2\) forms an ensemble class on which Tracy-Widom applies.

\(^2\)Student-t with 4 to 5 degrees of freedom.
The number of significant factors driving the U.S. equities and options market are as follow:

1. Equities in SPX: 15 significant factors (account for 55% of variance).

2. Options with underlying in SPX: 84 significant factors (account for 55% of variance).

3. All assets in OptionMetrics: 20 significant factors (account for 24% of variance).

4. All options with underlying asset in OptionMetrics: 108 significant factors (account for 50% of variance).