Universal bound on the measure of periodic windows of parameterized circle-maps.

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Abstract

One-parameter families f_t of circle diffeomorphisms are a common occurrence in dynamical systems. One subject of investigation is the variation of the rotation number with the parameter and how the parameter-range splits into periodic windows and a Cantor set of irrational rotation numbers. One of the earliest topics of investigation is how the measure of this Cantor set depends on the family, starting with the work of Arnold, Herman, etc. Studies of various parameterized circle maps seem to indicate that this measure approaches 1 when f_t is a small perturbation of the identity, and sometimes approaches 0 when f_t is close to a critical map. This paper describes a universal function η which gives an upper bound on the Lebesgue measure of the periodic windows based on the C^4 distance of f_t from the identity map. This confirms several observations made in the mathematical literature in the past.

1 Introduction and main results

In this paper, the unit circle S^1 will be identified as $\mathbb{R}/\mathbb{N}$, and $proj : \mathbb{R} \to \mathbb{N}$ is the associated quotient map. A homeomorphism of the circle $f : S^1 \to S^1$ can be lifted to a map $\bar{f} : \mathbb{R} \to \mathbb{R}$ under the covering map $proj$. It is well known (see for example, [1]) that the following limit exists and is a constant independent of z.

$$\rho(f) := \lim_{n\to\infty} \frac{\bar{f}(z) - z}{n}$$

This limit is called the **rotation number** of f. The rotation number is of fundamental importance in inferring the properties of the map and its limit points. If the rotation number is rational of the form $\frac{p}{q}$, then all points on S^1 is in the basin of attraction of some q-periodic point. On the other hand, if $\rho \notin \mathbb{Q}$, then f has the rotation $\theta \mapsto \theta + \rho \bmod 1$ as a factor map. In fact, a homeomorphism of the circle is conjugate to an irrational rotation iff it is transitive (see [2]). Dynamics on a torus $\mathbb{T}^d$ or on S^1 which are conjugate to an irrational rotation are called "quasiperiodic dynamics". They have interesting properties like transitivity, non-mixing, unique ergodicity, zero Lyapunov exponents etc. The existence of an invariant curve/manifold, on which the dynamics is transitive, can often lead to strong global properties. See for example, the role of blenders in [3, 4] and transversal quasiperiodic curves in [5].

An orientation preserving circle homeomorphism f is of the form given below.

$$f(\theta) = \theta + g(\theta) \bmod 1 \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is called the **periodic part** of the map F, and is periodic and of the same smoothness class as F. We are interested in parameterized families of C^3 circle diffeomorphisms, parameterized by a parameter $t \in [0, 1]$, which can be written similar to Eq. 1 in the following manner.

$$f_t : \theta \mapsto \theta + t + g_t(\theta) \bmod 1 \tag{2}$$

Here g is C^1 function of the parameter t and a C^3, 1-periodic function of θ.

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Partition of the parameter space. Given a parameterized family f_t, define $\boldsymbol{\mathcal{P}(f_t)}$ to be the set $\{t \in [0,1] : f_t$ has a periodic point$\}$, and by $\boldsymbol{\mathcal{Q}(f_t)}$ the set $\{t \in [0,1] : f_t$ is topologically conjugate to an irrational rotation$\}$. By Denjoy's theorem [6], a C^3 circle diffeomorphism with an irrational rotation number ρ is topologically conjugate to the rotation $\theta \mapsto \theta + \rho$ mod 1. Therefore,

$$[0,1] = \mathcal{P}(f_t) \sqcup \mathcal{Q}(f_t).$$

The focus of the paper will be on the following map $\boldsymbol{M}$ on 1-parameter families f_t.

$$M(f_t) := \mu(\mathcal{P}(f_t)). \tag{3}$$

$M(f_t)$ is the Lebesgue measure $\boldsymbol{\mu}$ of the set of parameter values t for which f_t has a periodic point. These set of values of t are often called the "mode-locked" regions [7] or "periodic windows". We will show later that M is an *upper semi-continuous* map. Brunovsky proved that there is a set of 1-parameter circle diffeomorphisms which is residual in the family of C^3-circle diffeomorphisms for which the periodic windows form an open, dense set in the parameter space $[0,1]$, (see Proposition 3 from [8]).

Arnold tongues and periodic windows. The dependence of the rotation number on the parameter t has been studied for over 50 years, see for example [1, 9] etc. If for some parameter value t_0, f_{t_0} has a stable periodic orbit with period n, then $\rho(f_t)$ is constant and $= k/n$ for t in some neighborhood of t_0. These intervals over which $\rho(f_t)$ is constant are called periodic windows. Arnold, in the seminal paper [9] studied the family

$$f_{t,\delta} : \theta \to \theta + t + \delta \sin(2\pi\theta) \text{ mod } 1$$

and proved that $M(f_t) \to 0$ as $\delta \to 0$. In his example, each of the countably infinitely many periodic windows shrink in width to a point at $\delta = 0$, as $\delta \to 0$, and monotonically thicken as δ is increased. The bifurcation diagrams of these windows with δ are called "Arnold tongues" because of their shape. The scaling laws of their width with parameter δ has been studied extensively in the general setting of $f_t \in \mathcal{F}$, in which $\sin(2\pi\theta)$ is replaced by some general periodic function $g_t(\theta)$.

The main result of this paper is to establish a bound on the total Lebesgue measure of the periodic windows, based on the C^4 norm of the periodic part g_t of the family.

Universality of Arnold tongues. Many universal properties of Arnold tongues have been observed in general families of the form Eq. 2. Cvitanovic et. al. [10] ordered the tongues based on the "Farey-sequence" ordering of the rationals and for all fixed values of t, found asymptotic scaling laws wrt their number in this ordering. Jonker [11] and Graczyk [7, 12] proved q^{-3} scaling laws, where q is the denominator of the rotation number. The differentiability properties of the boundaries of the Arnold tongues and their angle of contact at $\delta = 0$ has been studied in [13, 14].

Before stating the main theorem, a norm $\|\boldsymbol{f_t}\|_{\boldsymbol{\mathcal{F}}}$ will be defined on the space of C^4 parameterized circle diffeomorphisms.

$$\|f_t\|_{\mathcal{F}} := \max\left(\|g_t\|_{C^4}, \|\partial/\partial t g_t\|_{C^0}\right). \tag{4}$$

Note that $\|\frac{\partial}{\partial t}g_t\|_{C^0}$=max$\{ \left|\frac{\partial}{\partial t}g_t(\theta)\right| : t \in [0,1], \theta \in S^1 \}$. $\|g_t\|_{C^4}$ denotes the C^4 norm of g_t as a function of θ. Let $\text{Diff}^{\boldsymbol{r}}$ denote the family of C^r diffeomorphisms of S^1. We will define $\boldsymbol{\mathcal{F}}$ to be the set of parameterized circle diffeomorphismsf_t such that $f_t \in \text{Diff}^4$ for every $t \in [0,1]$ and such that $\|f_t\|_{\mathcal{F}} < 1$.

Theorem 1.1 *Let M and $\|\|_{\mathcal{F}}$ be as in Eq. 3 and Eq. 4. For every $r \in [0,1)$, there is an $\eta \in (0,1)$ such that if a parameterized family $f_t \in \mathcal{F}$ satisfies $|f_t|_{\mathcal{F}} \le r$, then $M(f_t) < \eta$.*

Remark. This theorem is consistent with the general observations made in the mathematical literature in the past on the gradual widening of the Arnold tongues as the periodic part g_t grows in C^4 norm. For example, the *standard family* $f_{t,\delta} : \theta \mapsto \theta + t + \delta(2\pi)^{-1}\sin(2\pi\theta)$ mod 1 of Arnold was shown to satisfy $M(f_{t,\delta}) \to 1$ as $\delta \to 0$ in [9], and $M(f_{t,\delta})) \to 0$ as $\delta \to 1$ in [15]. Theorem 1.1 is proved in Section 3.

2 Two lemmas

To prove Theorem 1.1, two important lemmas will be stated and proved.

The first lemma is a generalization of a lemma by Herman [3.8.2 [16]], which itself was an improvement of a theorem of Arnold's [Theorem 2, [9]]. The proof is based on ideas presented in [7.1. [16]] and has been provided for the sake of completeness and since no equivalent lemma has been found by the author in the mathematical literature.

Lemma 2.1 (Generalization of Herman's continuity theorem) *Let f_t be as in Eq. 2. Then for every $\epsilon > 0$, $\exists \delta > 0$ such that if g_t is a C^3 function and satisfies $\|g_t\|_{C^3} < \delta$, then $M(f_t) < 5\epsilon$.*

The lemma which is being generalized will be provided and used in the proof. Let for $\forall C > 0$, $\boldsymbol{\mathcal{D}(C)}$ denote the set $\{x \in [0,1] : \forall n \in \mathbb{N} - \{0\}, |e^{i2\pi nx} - 1| \geq C|n|^{-3}\}$. It is known (see for example, 7.1 [16]) that $\forall C > 0$, $\mathcal{D}(C)$ is compact and $\lim_{C\to 0}\mu\mathcal{D}(C) = 1$.

Lemma 2.2 (KAM theorem, [1]) *Let f be as in Eq. 1 and $C > 0$. Then $\exists K_0(C) > 0$ (with $K_0(C) \to 0$ as $C \to 0$) and $L(C) > 0$ (with $L(C) \to \infty$ as $C \to 0$) such that if the periodic part g of f is C^3 and $\|g\|_{C^3} = K \leq K_0$, then there is a continuous map $\lambda_g : \mathcal{D}(C) \to \mathbb{R}$ such that for every $s \in \mathcal{D}(C)$, there is a diffeomorphism $h_{g,s}$ of S^1 such that the following hold.*
(i) $\theta + \lambda_g(s) + g(\theta) \bmod 1 = h_{g,s}^{-1} \circ (\theta + s) \circ h_{g,s}$, i.e., the map $\theta \mapsto \theta + \lambda_g(s) + g(\theta) \bmod 1$ is conjugate via $h_{g,s}$ to a rotation by s,
(ii) $|\lambda_g(s) - s| \leq KL(C)$,
(iii) $|h_{f,s} - Id|_{C^0} + |dh_{f,s} - Id|_{C^0} \leq KL(C)$.
(iv) $\mu(\lambda_{g_t}(\mathcal{D}(C))) > 1 - \epsilon$.

Proof of Lemma 2.1. The version of the above lemma that is to be proven involves g_t instead of g. Let $\epsilon > 0$ be fixed for the rest of the proof. Let $C > 0$ be chosen so that $\mu(\mathcal{D}(C)) > 1 - \epsilon$. Let $K_0 = K_0(C)$ and $L = L(C)$ be as in Lemma 2.2 and let $K < K(C)$ be chosen so that $KL < \epsilon$. Then by Lemma 2.2, if $\|g_t\|_{C^3} < K$ then
(i) the map $\theta \mapsto \theta + \lambda_{g_t}(s) + g_t(\theta)$ is conjugate via a diffeomorphism $h_{g_t,s}$ to a rotation by s.
(ii)$\|h_{g_t,s} - Id\|_{C^1} \leq \epsilon$.

Let $\boldsymbol{\mathcal{D}'(C)}$:=$\{t \in \mathcal{D}(C) : t \in \lambda_{g_t}(\mathcal{D}(C))\}$. Note that if $t \in \mathcal{D}'(C)$, then $t = \lambda_{g_t}(s)$ for some $s \in (\mathcal{D}(C))$, so f_t is conjugate to a rotation by s. Therefore $\mathcal{P}(f) \cap \mathcal{D}'(C) = \varnothing$, so the lemma will be proved if it can be shown that $\mu(\mathcal{D}'(C)) > 1 - 5\epsilon$.

For every $s \in \mathcal{D}(C)$, let $\boldsymbol{\phi_s} : [0,1] \mapsto \mathbb{R}$ denote the map $t \mapsto \lambda_{g_t}(s)$. Each of the maps ϕ_s is continuous. To see this, note that h_{g_t} changes continuously with g_t (for proof, see for instance, Lemma 4, [17]). Therefore λ_{g_t} changes continuously with g_t and therefore with t.

We are interested in the fixed points of the graphs ϕ_s. For if $\exists t \in [0,1]$ such that $\phi_s(t) = t$, then $\lambda_{g_t}(s) = t$. So the map $f_t : \theta \mapsto \theta + t + g_t(\theta)$ is the same as $\theta \mapsto \theta + \lambda_{g_t}(s) + g_t(\theta)$ and is by definition, conjugate to a rotation by s, so $t \in \mathcal{D}'(C)$.

Let $\boldsymbol{\mathcal{D}''(C)} := \{s \in \mathcal{D}(C) :$ graph of ϕ_s has a fixed point$\}$. Since the maps ϕ_s are continuous and $\|\phi_s - s\|_{C^0} < \epsilon$, $[\epsilon, 1-\epsilon] \cap \mathcal{D}(C) \subseteq \mathcal{D}''(C)$. Let $\boldsymbol{\bar{\phi}(s)}$ denote this fixed point, $\forall s \in \mathcal{D}''(C)$. Note that $\phi_s(\bar{\phi}(s)) = \bar{\phi}(s)$, so $\bar{\phi}$ is a continuous function. By Claim (ii) of Lemma 2.2, $\|\phi_s - s\|_{C^0} < \epsilon$, therefore $\|\bar{\phi} - Id\|_{C^0} < \epsilon$.

The fixed points of ϕ_s is the image of $\mathcal{D}''(C)$ under $\bar{\phi}$. So proving that $\bar{\phi}(\mathcal{D}''(C)) > 1 - 5\epsilon$ is enough to prove the claim of the lemma.

Since $\bar{\phi}$ is a C^0 function satisfying $\|\bar{\phi} - Id\|_{C^0} < \epsilon$, if $A \subset [0,1]$ is a compact set, then $\mu\left(\bar{\phi}(A)\right) \geq (1-\epsilon)\mu(A)$. With this in mind, it is sufficient to prove that $\mu(\mathcal{D}''(C)) > 1 - 3\epsilon$. To this end, note that,

$$\mu(\mathcal{D}''(C)) \geq \mu([\epsilon, 1-\epsilon] \cap \mathcal{D}(C)) \geq \mu([\epsilon, 1-\epsilon]) + \mu(\mathcal{D}(C)) - \mu([\epsilon, 1-\epsilon] \cup \mathcal{D}(C)) \geq (1-2\epsilon) + (1-\epsilon) - (1) = 1 - 3\epsilon.$$

This completes the proof of Lemma 2.1. $\blacksquare$

Remark. Lemma 2.1 proves that the map M from Eq. 3 is continuous at $g_t \equiv 0$. The question of whether M is continuous everywhere is not known. However, for our purposes a notion weaker than continuity, called "semi-continuity" is needed.

Definition. Let X be a topological space, $f : X \to \mathbb{R}$. Then f is said to be **upper semi-continuous** at x if $\limsup_{y\to x} f(y) \le f(x)$. f is said to be upper semi-continuous if it is upper semi-continuous at every point $x \in X$. Equivalently, f is upper semi-continuous iff for every $a \in \mathbb{R}$, $f^{-1}(-\infty, a)$ is open. The following lemma proves that M from Eq. 3 is an upper semi-continuous function.

Lemma 2.3 (Semi-continuity lemma) *The map M defined in Eq. 3 is upper semi-continuous as a map from parameterized families of C^3-diffeomorphisms into $[0,1]$.*

Remark. Semi-continuity of the function M has been mentioned before in the literature but the author has found no proof of the claim, at least in the generality of the the problem being studied. Hence, it is being provided here.

Proof of Lemma 2.3. Let $f_t \in \mathcal{F}$, $\|f_t\|_{\mathcal{F}} = r < 1$ and $M(f_t) = \eta < 1$. It will be proved that M is upper semi-continuous at f_t. Fix $\epsilon > 0$, it will be shown that $\exists\delta > 0$ such that if $\left\|\hat{f}_t - f_t\right\|_{\mathcal{F}} < \delta$ for some $\hat{f}_t \in \mathcal{F}$, then $M(\hat{f}_t) < M(f_t) + 3\epsilon$.

Recall that $\mathcal{Q}(f_t)$ denotes the complement in $[0,1]$ of the set $\mathcal{P}(f_t)^C$. Following the idea in Proposition 6.2. of [16], divide the set $\mathcal{Q}(f_t)$ (up to a set of measure 0) as a union of nested compact sets $\cup_{k\in\mathbb{N}} D_k$, where $D_k := \{t : f_t$ is C^3 conjugate to the rotation by $\rho(f_t)$, via a conjugacy h_t satisfying $\|h_t\|_{C^3} \le k\}$. Therefore, $\lim_{k\to\infty} \mu(D_k) = 1 - \eta$.

Let $k \in \mathbb{N}$ be chosen large enough so that $\mu(D_k) \ge 1 - \eta - \epsilon$. Let $\hat{f}_t(\theta) = f_t(\theta) + \Delta_t(\theta)$ for some periodic perturbation term Δ_t. Let $\delta = k^{-2}\epsilon$ and $\|\Delta_t\|_{\mathcal{F}} < \delta$, so that $\left\|\hat{f}_t - f_t\right\|_{\mathcal{F}} < \delta$.

If $t_0 \in D_K$, then $f_{t_0} = h^{-1} \circ (\theta \mapsto \theta + \rho_0) \circ h$, where $\rho_0 = \rho(f_{t_0})$ and h is the conjugacy satisfying $\|dh_t\|_{C^0} \le k$.

$$\hat{f}_t : \theta \mapsto f_{t_0}(\theta) + \hat{\Delta}_t(\theta); \quad \hat{\Delta}_t(\theta) = (t - t_0) + \Delta_t(\theta) + f_t(\theta) - f_{t_0}(\theta).$$

Conjugating both sides by h gives

$$h \circ \hat{f}_t \circ h^{-1} : \theta \mapsto \theta + \rho_0 + \nu_t(\theta)$$

Since $\|dh_t\|_{C^0} \le k$, $\|\nu_t\|_{C^3} \le k^2\|\Delta_t\|_{C^3} < \epsilon$. By Lemma 2.1, a continuous function $\Psi_\Delta : D_k \to \mathbb{R}$ may be constructed so that if $t' = \Psi_\Delta(t_0)$, then $|t_0 - t'| < \epsilon$ and the map

$$\theta \mapsto \theta + \rho_0 + \nu_{t'}(\theta)$$

from above is conjugate to an irrational rotation. Therefore, similar to the proof of Lemma 2.1,

$$\mu(\mathcal{Q}(\hat{f}_t)) > (1 - \epsilon)\mu(D_K) > (1 - \epsilon)(1 - \eta - \epsilon)) > 1 - \eta - 3\epsilon,$$

so $M(\hat{f}_t) < \eta + 3\epsilon$. This completes the proof of Lemma 2.3. $\blacksquare$

3 Proof of Theorem 1.1

Let $\boldsymbol{\eta}$: $[0,1] \to [0,1]$ be defined as

$$\forall r \in [0,1],\ \eta(r) := \sup\{\mu\mathcal{P}(f) \ : \ f \in \mathcal{F},\ \|f_t\|_{\mathcal{F}} < r\}.$$

Note that η is a non-decreasing function of r. Moreover, $\eta(0) = 0, \eta(1) = 1$.

Claim. η is a right-continuous function. The proof will be by contradiction, so let η not be right-continuous at some $r \in (0,1)$. Since η is non-decreasing, this means that $\exists \delta > 0$ and $f_{n,t}$ a sequence of parameterized families in $\mathcal{F}$ such that the norms $\|f_{n,t}\|_{\mathcal{F}}$ is monotonically decreasing and converges to r^+, but $M(f_{n,t}) \geq \eta(r) + \delta$. Since their periodic parts $g_{n,t}$ is a bounded sequence in $C^4(S^1)$, they have a limit point g_t in $C^3(S^1)$.

Let $f_t : \theta \mapsto \theta + t + g_t(\theta)$. By the upper semi-continuity of M (Lemma 2.3), $M(f_t) \geq \limsup_{n\to\infty} M(f_{n,t}) \geq \eta(r) + \delta$. However, $\|f_t\|_{\mathcal{F}}$ must equal r, so by the definition of η, $\eta(r) \geq M(f_t)$, which is a contradiction. So the assumption that η is not right-continuous at r was wrong, so the claim is proved.

Claim. η is a left-continuous function. To see this, first note that by the upper semi-continuity of M,

$$\limsup_{r' \to r^-} \eta(r') \leq \eta(r).$$

So if η is not left-continuous at $r \in (0,1)$, then there exists $\delta > 0$ such that

$$\limsup_{r' \to r^-} \eta(r') < \eta(r) - \delta.$$

So there must exist $f_t \in \mathcal{F}$ such that $\|f_t\|_{\mathcal{F}} = r$ and $M(f_t) = r$. Now by adding a small perturbation to g_t over a range of parameter values of length $< 0.5\delta$, it is possible to get a new family $\bar{f}_t$ such that $\|\bar{f}_t\|_{\mathcal{F}} < r$. However, $M(\bar{f}_t)$ could only have decreased by 0.5δ, so $M(\bar{f}_t) > \eta(r) - \delta$. This contradicts the equation above. This completes the proof of the claim.

Claim. η is a continuous function. This follows from the previous two claims.

Claim $\eta(r) < 1$ for $r \in (0,1)$. To prove this, define the universal constant $\boldsymbol{r_*}$ as $r_* := \inf\{r \in [0,1] : \eta(r) = 1\}$. Lemma 2.1 proves that $r_* > 0$. Suppose $r^* < 1$. Then there exists a sequence $f_{n,t}$ of parameterized families in $\mathcal{F}$ such that the norms $\|f_{n,t}\|_{\mathcal{F}}$ is monotonically decreasing and converges to r_*^+, but $M(f_{n,t}) = 1$. Since $g_{n,t}$ is a bounded sequence in $C^4(S^1)$, they have a limit point g_t in $C^3(S^1)$.

Let $f_t := (\theta) \mapsto \theta + t + g_t(\theta)$. By the upper semi-continuity of M (Lemma 2.3), $1 \geq M(f_t) \geq \limsup_{n\to\infty} M(f_{n,t}) = 1$. This contradicts Theorem 6.1. from [16], which states that a family f_t with a non-constant rotation number cannot have a full Lebesgue measure set of parameters with rational rotation number.

This completes the proof of the claim and also of Theorem 1.1. ∎

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