# Time averaging for flows

$\Phi^t$ is a $C^0$ flow on a space X with invariant ergodic measure $\mu$. $U^t$ is the Koopman group. $A_T : L^2 \to L^2 := (A_T f)(x) = T^{-1}\int_{[0,T]} f(\Phi^t x)dx$. $H_T$:= range$(A_T)$. $V$ is the generator of $U^t$.

The purpose of this write-up is to show that range$(A_T) \subseteq$ dom$(V)$. In other words, $V \circ A_T$ is well defined on the entire space $L^2(X,\mu)$.

$L$ is the limiting operator $\lim_{t\to 0} A_t$, Dom$(L)$ is the subset of $L^2(X,\mu)$ on which $L$ can be defined.

1. Time averaging operator is a Markov operator : $A_T 1 = 1$, $\|A_T f\| \leq \|f\|$ $\forall f \in L^2$.

2. Time averaging commutes with Koopman : $A_T U^t = U^t A_T$, $\forall t \in \mathbb{R}$.

3. Averaging preserves Koopman-invariant subspaces : Subspaces of $L^2(X,\mu)$ which are invariant under $U^t$ are also invariant under $A_T$.

4. Difference operator formula : $t^{-1}(U^t - Id)A_T f = T^{-1}A_t(U^T - Id)f$.
   Proof : Note that $(U^t - Id)A_T f = T^{-1}\left[\int_{[T,T+t]} f(\Phi^t x)dx - \int_{[0,t]} f(\Phi^t x)dx\right] = \frac{t}{T}A_t(U^T - Id)f$.

5. dom$(V \circ A_T)$ = dom$(L)$ and these are invariant under $U^t$.
   Proof : It will be first shown that dom$(V \circ A_T)$ and dom$(L)$ are invariant under $U^t$.
   Invariance of dom$(V \circ A_T)$ :
   Invariance of dom$(L)$ :
   The claim will be proved by showing that the two domains are subsets of each other.
   dom$(V\circ A_T) \subseteq$ dom$(L)$ : Let $f \in$ dom$(V\circ A_T)$. Then by "Difference operator formula", $f \in$ dom$(L)\circ(U^T - Id)$.
   The range of the operator $U^T - Id$ is called the set of "coboundaries" and these are dense in $L^2(X,\mu)$. ...
   dom$(V \circ A_T) \supseteq$ dom$(L)$ : Let $f \in$ dom$(L)$. Then since dom$(L)$ is invariant under the Koopman group, $(U^T - Id)f \in dom(L)$ and therefore, by "Difference operator formula", $f \in$ dom$(V \circ A_T)$.
   This completes the proof of the claim.

6. Let $\mathcal{C}$:= $\{\ f \in L^2(X,\mu) : t \mapsto \langle U^t f, f\rangle$ is continuous at $t = 0\}$. Then $\mathcal{C} \subset$ dom$(L)$ and $\forall f \in \mathcal{C}$, $Lf = f$.
   Proof : Let $f \in \mathcal{C}$, so there is a function $\epsilon : \mathbb{R} \to \mathbb{C}$ such that $\lim_{t\to 0}\epsilon(t) = 0$ and $\langle U^t f, f\rangle = \|f\|^2 + \epsilon(t)$. Then $\langle A_t f, f\rangle = \int_X t^{-1}\int_{[0,t]}[f(\Phi^s x)]ds\bar{f}(x)d\mu(x) = t^{-1}\int_{[0,t]}\left[\int_X f(\Phi^s x)\bar{f}(x)d\mu(x)\right]ds = \|f\|^2_{L^2} + t^{-1}\int_{[0,t]}\epsilon(s)ds$, which $\to \|f\|^2_{L^2}$ as $t \to 0$.
   Secondly, $\langle A_t f, A_t f\rangle = t^{-2}\int_{r,s\in[0,t]}\left[\int_X f(\Phi^s x)\overline{f(\Phi^r x)}d\mu(x)\right]drds = t^{-2}\int_{r,s\in[0,t]}\left[\int_X f(\Phi^{s-r}x)\overline{f(x)}d\mu(x)\right]drds =$ $t^{-2}\int_{r,s\in[0,t]}\left[\|f\|^2_{L^2} + \epsilon(s-r)\right]drds = \|f\|^2_{L^2} + t^{-2}\int_{r,s\in[0,t]}\epsilon(s-r)drds$, which converges to $\|f\|^2_{L^2}$ as $t \to 0$.
   Therefore, $\|A_t f - f\|^2_{L^2} = \langle A_t f - f, A_t f - f\rangle = \langle A_t f, A_t f\rangle + \|f\|^2_{L^2}$ - $\langle A_t f, f\rangle$ - $\langle f, A_t f\rangle$, which $\to 0$ as $t \to 0$.
   Thus, $Lf = \lim_{t\to 0} A_t f = f$.

7. It will now be shown that $\mathcal{C} = L^2(X,\mu)$ through a series of claims.

   7.1 $\mathcal{C}$ is closed under $U^t$.
   Proof : This follows from the fact that $U^t$ is a unitary group.

   7.2 If $\mu(X) < \infty$, then $L^\infty(X) \subseteq \mathcal{C}$. In particular, if $X$ is compact, then $C^0(X) \subseteq \mathcal{C}$.
   Proof : Let $f \in L^\infty(X)$ and $M < \infty$ be the essential supremum of $f$. Then $\forall t \in \mathbb{R}$, $\|U^t f\|_{L^\infty} = M$. Then by the dominated convergence theorem, $t \mapsto \langle U^t f, f\rangle$ is continuous at $t = 0$ and therefore, $f \in \mathcal{C}$.

   7.3 If $\mu(X) < \infty$, then $\mathcal{C} = L^2(X,\mu)$.
   Proof : Let $f \in L^2(X,\mu)$ and $\epsilon > 0$. WLOG, $\|f\|_{L^2} = 1$. It will be shown that for $t$ sufficiently small, $\left|\langle U^t f, f\rangle - \|f\|^2_{L^2}\right| < 5\epsilon$. There is an $M_\epsilon > 0$ such that if $X_\epsilon$:= $\{\ |f| > M_\epsilon\ \}$, $f_\epsilon$:= $1_{X_\epsilon} f$ and $g_\epsilon$:= $f - f_\epsilon$, then $\|g_\epsilon\|^2_{L^2} < \epsilon$. the claim will be proved by bounding each of the terms in the RHS of the following equation.

$$\begin{aligned}\langle U^t f, f\rangle - \|f\|^2_{L^2} &= \langle U^t(f_\epsilon + g_\epsilon), f_\epsilon + g_\epsilon\rangle - \|f\|^2_{L^2} \\ &= \langle U^t g_\epsilon, U^t g_\epsilon\rangle + \langle U^t g_\epsilon, f_\epsilon\rangle + \langle U^t f_\epsilon, g_\epsilon\rangle + \left[\langle U^t f_\epsilon, f_\epsilon\rangle - \|f_\epsilon\|^2_{L^2}\right] + \left[\|f_\epsilon\|^2_{L^2} - \|f\|^2_{L^2}\right]\end{aligned}$$

   Firstly, $|\langle U^t g_\epsilon, U^t g_\epsilon\rangle| = \|g_\epsilon\|^2_{L^2} \leq \epsilon$.
   Secondly, $|\langle U^t f_\epsilon, g_\epsilon\rangle| = \left|\int_{X-X_\epsilon}(U^t f)(x)g_\epsilon(x)d\mu(x)\right| \leq \int_{X-X_\epsilon}|(U^t f)(x)||g_\epsilon(x)|d\mu(x) \leq M_\epsilon\int_{X-X_\epsilon}|g_\epsilon(x)|d\mu(x)$ $\leq \int_{X-X_\epsilon}|g_\epsilon(x)|^2 d\mu(x) \leq \|g_\epsilon\|^2_{L^2} \leq \epsilon$.
   Similarly, $|\langle U^t g_\epsilon, f_\epsilon\rangle| \leq \epsilon$.
   By construction, $\left|\|f_\epsilon\|^2_{L^2} - \|f\|^2_{L^2}\right| = \|g_\epsilon\|^2_{L^2} \leq \epsilon$.
   Finally, since $f_\epsilon \in L^\infty(X)$, $\exists\delta > 0$ such that $\forall t \in \mathbb{R}$ such that $|t| < \delta$, $\left|\langle U^t f_\epsilon, f_\epsilon\rangle - \|f_\epsilon\|^2_{L^2}\right| < \epsilon$.
   Therefore, $\forall|t| < \delta$, $\left|\langle U^t f, f\rangle - \|f\|^2_{L^2}\right| < 5\epsilon$.

8. Therefore, if $\mu(X) < \infty$, then $V \circ A_T$ is defined on all of $L^2(X,\mu)$.