Time averaging for flows

\( \Phi^t \) is a \( C^0 \) flow on a space \( X \) with invariant ergodic measure \( \mu \). \( U^t \) is the Koopman group. \( A_T : L^2 \to L^2 : = (\mathcal{A}_T f)(x) = T^{-1} \int_{[0, t]} f(\Phi^x) dx \). \( H_T := \text{range}(A_T) \). \( V \) is the generator of \( U^t \).

The purpose of this write-up is to show that \( \text{range}(A_T) \subseteq \text{dom}(V) \). In other words, \( V \circ A_T \) is well defined on the entire space \( L^2(X, \mu) \).

\( L \) is the limiting operator \( \lim_{t \to 0} A_t \), \( \text{Dom}(L) \) is the subset of \( L^2(X, \mu) \) on which \( L \) can be defined.

1. Time averaging operator is a Markov operator: \( A_T f \)

2. Time averaging commutes with Koopman: \( A_T U^t = U^t A_T, \forall t \in \mathbb{R} \)

3. Averaging preserves Koopman-invariant subspaces: Subspaces of \( L^2(X, \mu) \) which are invariant under \( U^t \) are also invariant under \( A_T \).

4. Difference operator formula: \( t^{-1}(U^t - Id)A_T f = T^{-1} A_t (U^t - Id) f \).

   Proof: Let \( (\mathcal{A}_T f)(x) = \int_{[0, t]} f(\Phi^x) dx \) \( \in C \). Secondly, \( \int_{[0, t]} f(\Phi^x) dx \) \( \in L^2(\mathcal{A}_T f) \).

   Therefore, \( \| f \|_2 \leq \| \tilde{\mathcal{A}}_T f \|_2 \). Thus, \( \| f \|_2 = \| \mathcal{A}_T f \|_2 \).

   Thus, \( Lf \) is continuous at \( f = 0 \) as \( t \to 0 \).

5. Proof: \( \| f \|_2 \) is continuous at \( f = 0 \) as \( t \to 0 \).

   Therefore, \( \| f \|_2 \) is continuous at \( f = 0 \) as \( t \to 0 \).

   Thus, \( \| f \|_2 = \| \mathcal{A}_T f \|_2 \).

6. Let \( C := \{ f \in L^2(X, \mu) : t \mapsto \langle f, f \rangle \) is continuous at \( t = 0 \} \). Then \( C \subseteq \text{Dom}(L) \). Let \( f \in C \).

   Proof: \( f \in C \) such that \( \lim_{t \to 0} \| f \|_2 = \| f \|_2 \).

   Therefore, \( \| f \|_2 \) is continuous at \( t = 0 \).

7. It will now be shown that \( C = L^2(X, \mu) \) through a series of claims.

7.1 \( C \) is closed under \( U^t \).

   Proof: From the fact that \( U^t \) is a unitary group.

7.2 If \( \mu(X) < \infty \), then \( L^\infty(X) \subseteq C \). In particular, if \( X \) is compact, then \( C^0(X) \subseteq C \).

   Proof: \( f \in L^\infty(X) \) and \( M < \infty \) be the essential supremum of \( f \). Then \( \forall t \in \mathbb{R} \), \( \| U^t f \|_\infty = M \). Then by the dominated convergence theorem, \( t \mapsto \langle U^t f, f \rangle \) is continuous at \( t = 0 \) and therefore, \( f \in C \).

7.3 If \( \mu(X) < \infty \), then \( C = L^2(X, \mu) \).

   Proof: \( f \in L^2(X, \mu) \) and \( \epsilon > 0 \). WLOG, \( \| f \|_2 = 1 \). It will be shown that for \( t \) sufficiently small, \( \| U^t f, f \|_2 < \epsilon \).

   Firstly, \( \| U^t f, f \|_2 = \| f \|_2 \). Then \( \forall t \in \mathbb{R} \), \( \| f \|_2 = 1 \). Then by the dominated convergence theorem, \( t \mapsto \langle U^t f, f \rangle \) is continuous at \( t = 0 \) and therefore, \( f \in C \).

8. Therefore, if \( \mu(X) < \infty \), then \( V \circ A_T \) is defined on all of \( L^2(X, \mu) \).