Let $M$ be an $d$-dimensional manifold equipped with the the Borel $\sigma$-algebra $B$. $\Phi^t$ is a continuous flow on $M$ with a compact invariant set $X \subset M$, which supports an invariant ergodic measure $\mu$. Further, let $X$ be an attractor, i.e., there is a neighborhood $U$ of $X$ such that (i) for every $t > 0$, $\Phi^t(U) \subset U$; and (ii) $X = \cap_{t=0}^{\infty} \Phi^t(X)$.

**Physical measures.** The set $C^0(X)$ of continuous functions with the uniform topology forms a separable Banach space, and the set $M$ of finite, complex-valued measures form a subspace of dual space $X^0(U)$.* Given $\mu \in M$, point $x \in M$ is said to be $\mu$-generic if the following holds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi^nx) = \int_X f(y) d\mu(y), \quad \forall f \in C^0(U).$$  \hspace{1cm} (1)

The measure $\mu$ is said to be a *physical measure* if the set of $\mu$-generic points has positive Lebesgue measure. A more generalized notion of physical measures is needed for Fourier averages. Thus $\mu$-generic points are initial points $x_0 \in M$ along which the trajectory averages in (1) of continuous functions converge to their ergodic averages.

**The Koopman operator.** Koopman operators act on observables by time-shifts. The space $L^2(X, B, \mu)$ of square-integrable functions on $X$ will be our space of observables / measurements. Given an observable $f : X \to \mathbb{C}$ and time $t \in \mathbb{R}$, let $U^t : L^2 \to L^2$ be the operator defined as

$$(U^t f) : x \mapsto f(\Phi^tx).$$

$U^t$ is called the associated Koopman operator associated to the flow, at time $t$. since $\Phi^t$ is a invertible map, $U^t$ is a unitary map and therefore has all of its eigenvalues on the unit circle of the complex plane. An eigenfunction $z$ of $U^t$ satisfies the following equation for some $\omega \in \mathbb{R}$.

$$U^t z = \exp(i\omega t) z \hspace{1cm} (2)$$

Under the assumption of ergodicity in , for every $\omega \in \mathbb{R}$, the eigenspace of $U^t$ corresponding to the eigenvalue $\exp(i\omega t)$ is of dimension at most 1. Hence, once can define $z_\omega$ to be 0 if the eigenspace is $\{0\}$, or to be any unit norm eigenfunction. For every $\omega \in \mathbb{R}$ and every $f \in L^2(X, \mu)$, there are the following projection operators.

$$\mu_\omega(f) := \langle f, z_\omega \rangle_{L^2(X, \mu)},$$

$$\pi_\omega(f) := \langle f, z_\omega \rangle_{L^2(X, \mu)} z_\omega = \mu_\omega(f) z_\omega. \hspace{1cm} (3)$$

$$A_{\omega,N} := \frac{1}{N} \sum_{n=0}^{N-1} e^{-i\omega n} U^n. \hspace{1cm} (4)$$

**Lemma 0.1** $A_{\omega,N}$ converges to $\pi_\omega$ pointwise in $L^2(X, \mu)$.

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Generalized notion of genericity. \( x \in M \) is said to be \((\omega, \mu)\)-generic if the following holds

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(-i\omega n) f(\Phi^n \Delta t x) = \int_X f(y) d\mu(y), \quad \forall f \in C^0(\mathcal{U}).
\]

(5)

For each \( \omega \in \mathbb{R} \) and \( x \in \mathcal{U} \), one can define a family of sampling measures as follows.

\[
\delta_{\omega, N, x} := \frac{1}{N} \sum_{n=0}^{N-1} \exp(-i\omega n) \delta_{x_n}; \quad x_n = \Phi^n \Delta t x_0.
\]

(6)

Then one has the following way to characterize \((\omega, \mu)\)-genericity.

**Lemma 0.2** A point \( x \in \mathcal{U} \) is \((\omega, \mu)\)-generic iff \( \delta_{\omega, N, x} \) converges to \( e_x \circ \mu_{\omega} \) in the \( C^0(\mathcal{U})^* \) topology.

Let \( \forall x \in M, e_x \in C^0(\mathcal{U})^* \) be the point evaluation function at \( x \), i.e., for every \( f \in C^0(\mathcal{U}) \), \( e_x(f) = f(x) \). Then,

\[
\delta_{\omega, N, x} = e_x \circ A_{\omega, N}
\]

(7)

**Question 1** Does \( \mu \)-genericity imply \((\omega, \mu)\)-genericity for every \( \omega \in \mathbb{R} \)? Are there counter examples known?

**Question 2** What conditions would guarantee \((\omega, \mu)\)-genericity for

(i) every \( \omega \in \mathbb{R} \) ?

(ii) some \( \omega \) in real ?