Generalized notion of genericity

Let M be an d-dimensional manifold equipped with the the Borel σ-algebra $\mathcal{B}$. Φ^t is a continuous flow on M with a compact invariant set $X \subset M$, which supports an invariant ergodic measure μ. Further, let X be an attractor, i.e., there is a neighborhood $\mathcal{U}$ of X such that (i) for every $t > 0$, $\Phi^t(\bar{U}) \subset U$; and (ii) $X = \cap_{t=0}^{\infty} \Phi^t(X)$.

Physical measures. The set $C^0(X)$ of continuous functions with the uniform topology forms a separable Banach space, and the set $\mathcal{M}$ of finite, complex-valued measures form a subspace of dual space $X^0(X)^*$. Given $\mu \in \mathcal{M}$, point $x \in M$ is said to be μ-generic if the following holds.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi^{n\,\Delta t} x) = \int_X f(y)\, d\mu(y), \quad \forall f \in C^0(\mathcal{U}). \tag{1}$$

The measure μ is said to be a *physical measure* if the set of μ-generic points has positive Lebesgue measure. A more generalized notion of physical measures is needed for Fourier averages. Thus μ-generic points are initial points $x_0 \in M$ along which the trajectory averages in (1) of continuous functions converge to their ergodic averages.

The Koopman operator. Koopman operators act on observables by time-shifts. The space $L^2(X, \mathcal{B}, \mu)$ of square-integrable functions on X will be our space of observables / measurements. Given an observable $f : X \to \mathbb{C}$ and time $t \in \mathbb{R}$, let $U^t : L^2 \to L^2$ be the operator defined as

$$(U^t f) : x \mapsto f\left(\Phi^t x\right).$$

U^t is called the associated Koopman operator associated to the flow, at time t. since Φ^t is a invertible map, U^t is a unitary map and therefore has all of its eigenvalues on the unit circle of the complex plane. An eigenfunction z of U^t satisfies the following equation for some $\omega \in \mathbb{R}$.

$$U^t z = \exp(i\omega t) z \tag{2}$$

Under the assumption of ergodicity in , for every $\omega \in \mathbb{R}$, the eigenspace of U^t corresponding to the eigenvalue $\exp(i\omega t)$ is of dimension at most 1. Hence, once can define z_ω to be 0 if the eigenspace is $\{0\}$, or to be any unit norm eigenfunction. For every $\omega \in \mathbb{R}$ and every $f \in L^2(X, \mu)$, there are the following projection operators.

$$\begin{aligned} \mu_\omega(f) &:= \langle f, z_\omega \rangle_{L^2(X,\mu)}. \\ \pi_\omega(f) &:= \langle f, z_\omega \rangle_{L^2(X,\mu)} z_\omega = \mu_\omega(f) z_\omega. \end{aligned} \tag{3}$$

$$\mathcal{A}_{\omega,N} := \frac{1}{N} \sum_{n=0}^{N-1} e^{-i\omega n} U^n. \tag{4}$$

Lemma 0.1 *$\mathcal{A}_{\omega,N}$ converges to π_ω pointwise in $L^2(X, \mu)$.*

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Generalized notion of genericity. $x \in M$ is said to be (ω, μ)-generic if the following holds

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(-i\omega n) f(\Phi^{n\,\Delta t} x) = \int_X f(y)\, d\mu(y), \quad \forall f \in C^0(\mathcal{U}). \tag{5}$$

For each $\omega \in \mathbb{R}$ and $x \in \mathcal{U}$, one can define a family of sampling measures as follows.

$$\delta_{\omega,N,x} := \frac{1}{N} \sum_{n=0}^{N-1} \exp(-i\omega n) \delta_{x_n}; \quad x_n = \Phi^{n\Delta t} x_0. \tag{6}$$

Then one has the following way to characteriz (ω, μ)-genericity.

Lemma 0.2 *A point $x \in \mathcal{U}$ is (ω, μ)-generic iff $\delta_{\omega,N,x}$ converges to $e_x \circ \mu_\omega$ in the $C^0(\mathcal{U})^*$ topology.*

Let $\forall x \in M$, $e_x \in C^0(\mathcal{U})^*$ be the point evaluation function at x, i.e., for every $f \in C^0(\mathcal{U})$, $e_x(f) = f(x)$. Then,

$$\delta_{\omega,N,x} = e_x \circ \mathcal{A}_{\omega,N} \tag{7}$$

Question 1 *Does μ-genericity imply (ω, μ)-genericity for every $\omega \in \mathbb{R}$? Are their counter examples known ?*

Question 2 *What conditions would guarantee (ω, μ)-genericity for*
(i) every $\omega \in \mathbb{R}$?
(ii) some ωinreal ?